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# Odd-Graceful Labelings of Trees of Diameter 5

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**Abstract**

A difference vertex labeling of a graph $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ the weight $|f(x) - f(y)|$. A difference vertex labeling $f$ of a graph $G$ of size $n$ is odd-graceful if $f$ is an injection from $V(G)$ to $\{0, 1, ..., 2n - 1\}$ such that the induced weights are $\{1, 3, ..., 2n - 1\}$. We show here that any forest whose components are caterpillars is odd-graceful. We also show that every tree of diameter up to five is odd-graceful.

**Keywords:** Odd-graceful labeling, $\alpha$-labeling, trees of didmeter 5.

**2000 Mathematics Subject Classification:** Primary 05C78

## 1. Introduction

Let $G$ be a graph of order $m$ and size $n$, a *difference vertex labeling* of $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ a label or *weight* given by the absolute value of the difference of its vertex labels. Graceful labelings are a well-known type of difference vertex labeling; a function $f$ is a *graceful labeling* of a graph $G$ of size $n$ if $f$ is an injection from $V(G)$ to the set $\{0, 1, ..., n\}$ such that, when each edge $xy$ of $G$ has assigned the weight $|f(x) - f(y)|$, the resulting weights are distinct; in other words, the set of weights is $\{1, 2, ..., n\}$. A graph that admits a graceful labeling is said to be *graceful*.

When a graceful labeling $f$ of a graph $G$ has the property that there exists an integer $\lambda$ such that for each edge $xy$ of $G$ either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$, $f$ is named an $\alpha$-labeling and $G$ is said to be an $\alpha$-graph. From the definition it is possible to deduce that an $\alpha$-graph is necessarily bipartite and that the number $\lambda$ (called the *boundary value* of $f$) is the smaller of the two vertex labels that yield the edge with weight 1. Some examples of $\alpha$-graphs are the cycle $C_n$ when $n \equiv 0 \pmod 4$, the complete bipartite graph $K_{m,n}$, and caterpillars (i.e., any tree with the property that the removal of its end vertices leaves a path).

A little less restrictive than $\alpha$-labelings are the odd-graceful labelings introduced by Gnanajothi in 1991 [4]. A graph $G$ of size $n$ is *odd-graceful* if there is an injection $f : V(G) \rightarrow \{0, 1, 2, ..., 2n-1\}$ such that the set of induced weights is $\{1, 3, ..., 2n-1\}$. In this case, $f$ is said to be an *odd-graceful labeling* of $G$. One of the applications of these labelings is that trees of size $n$, with a suitable odd-graceful labeling, can be used to generate cyclic decompositions of the complete bipartite graph $K_{n,n}$. In Figure 1 we show an odd-graceful tree of size 6 together with its embedding in the circular arrangement used to produce the cyclic decomposition of $K_{6,6}$. Once the labeled tree has been embedded, successive 60° (counterclockwise) rotations produce the desired cyclic decomposition of $K_{6,6}$.

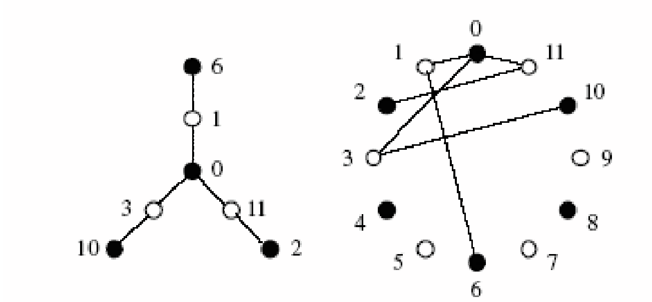


Figure 1: Cyclic decomposition of $K_{6,6}$

Gnanajothi [4] proved that the class of odd-graceful graphs lies between the class of $\alpha$-graphs and the class of bipartite graphs; she proved that every $\alpha$-graph is also odd-graceful. The reverse case does not work, for example the odd-graceful tree shown in Figure 1 is the smallest tree without an $\alpha$-labeling. Since many families of $\alpha$-graphs are known, the most attractive examples of odd-graceful graphs are those without an $\alpha$-labeling or where an $\alpha$-labeling is unknown; for instance, Gnanajothi [4] proved that the following are odd-graceful graphs: $C_n$ when $n \equiv 2 \pmod 4$, the disjoint union of $C_4$, the prism $C_n \times K_2$ if and only if $n$ is even, and trees of diameter 4 among others. Eldergil [2] proved that the one-point union of any number of copies of $C_6$ is odd-graceful. Seoud, Diab, and Elsakhawi [5] showed that a connected $n$-partite graph is odd-graceful if and only if $n = 2$ and that the join of any two connected graphs is not odd-graceful.

A detailed account of results in the subject of graph labelings can be found in Gallian' survey [3].

Gnanajothi [4] conjectured that all trees are odd-graceful and verified this conjecture for all trees with order up to 10. The author has extended this up to trees with order up to 12[1]. In this paper we prove that all trees of diameter 5 are odd-graceful and that any forest whose components are caterpillars is odd-graceful.

[1] Odd-graceful labelings of trees of order 11 and 12 can be found at http://cims.clayton.edu/cbarrien/research

## 2. Odd-Graceful Forests

In this section we study *forests* that accept odd-graceful labelings. Recall that a forest with more than one component cannot be graceful because it has "too many vertices". Gnanajothi [4] proved that every $\alpha$-graph is odd-graceful. In fact, let $G$ be an $\alpha$-graph of size $n$. Suppose that $f$ is an $\alpha$-labeling of $G$ such that $\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}$, where $\{A, B\}$ is the bipartition of $V(G)$. An odd-graceful labeling of $G$ is given by

$$g(x) = \begin{cases} 2f(x), & x \in A \\ 2f(x) - 1, & x \in B. \end{cases}$$

In Figure 2 we show an example of an $\alpha$-labeling of a caterpillar of size 10, together with its corresponding odd-graceful labeling. We use these labelings in the proof of Theorem 1.

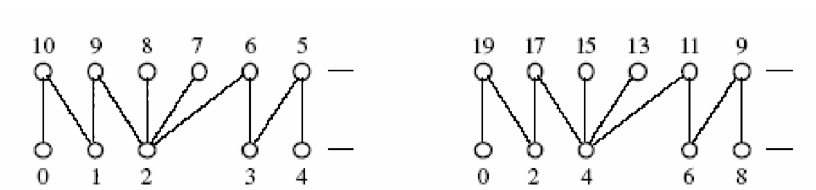


Figure 2: Odd-graceful labeling of a caterpillar

**Theorem 1.** *Any forest whose components are caterpillars is odd-graceful.*

*Proof.* Let $F_i$ be a caterpillar of size $n_i \geq 1$, for $1 \leq i \leq k$. Let $u_i, v_i \in V(F_i)$ such that $d(u_i, v_i) = diam(F_i)$; so identifying $v_i$ with $u_{i+1}$, for each $1 \leq i \leq k-1$, we have a caterpillar $F$ of size $\sum_{i=1}^{k} n_i = n$. Now we proceed to find both, the $\alpha$-labeling of $F$ and its corresponding odd-graceful labeling, using the scheme shown in Figure 2. Once the odd-graceful labeling has been obtained, we disengage each caterpillar $F_i$ from $F$, keeping their labels; in this form, the weights induced are $\{1, 3, ..., 2n-1\}$. To eliminate the overlapping of labels we subtract 1 from each vertex label of $F_i$ when $i$ is even, in this way the weights remain the same and the labels assigned on $u_{i+1}$ and $v_i$ differ by one unit. Therefore, the labeling of the forest $\bigcup_{i=1}^{k} F_i$ is odd-graceful. □

In Figure 3 we show an example of this construction using the odd-graceful labeling obtained in Figure 2.

The procedure used in this proof can be extended to the disjoint union of graphs with $\alpha$-labelings. In fact, suppose that the concatenation of blocks $B_1, B_2, ..., B_k$ results in a graph $G$ whose block-cutpoint graph is a path; in [1] we proved that $G$ is an $\alpha$-graph

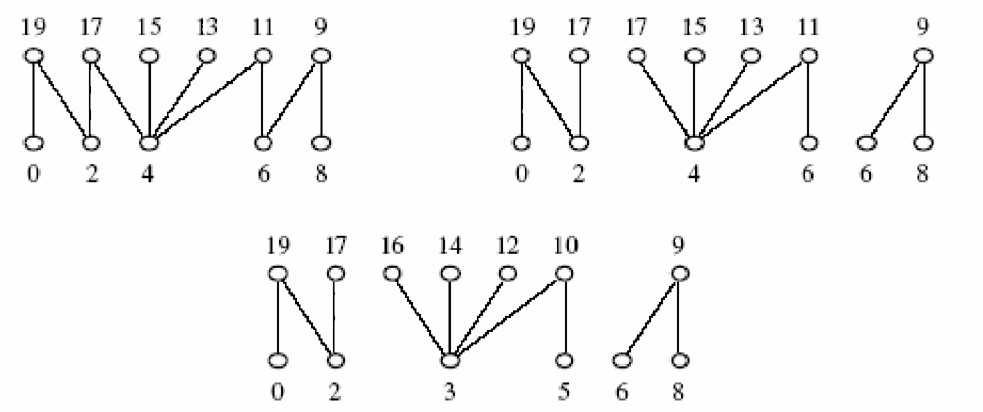


Figure 3: Odd-graceful labeling of a forest

provided that each $B_i$ is an $\alpha$-graph. Transforming this $\alpha$-labeling into an odd-graceful labeling and disconnecting $G$ into blocks, the disjoint union of these blocks is odd-graceful.

**Theorem 2.** *The disjoint union of blocks that accept $\alpha$-labelings is odd-graceful.*

As consequence, any forest which components are $\alpha$-trees is odd-graceful.

## 3. Odd-Graceful Trees of Diameter Five

Every tree of diameter at most 3 is a caterpillar, therefore it is odd-graceful. Gnanajothi [4] proved that every rooted tree of height 2 (that is, diameter 4) is odd-graceful. In the next theorem, we represent trees of diameter 5 as rooted trees of height 3 and prove that they are odd-graceful.

Let $T$ be a tree of diameter 5; $T$ can be represented as a rooted tree of height 3 by using any of its two central vertices as the root vertex. Note that only one of the vertices in level 1 has descendants in level 3; this vertex will be located in the right extreme of level 1. Now, within each level, the vertices are placed from left to right in such a way that their degrees are increasing. In the proof of the next theorem we use this type of representation of $T$, that is, assuming that $v$ (one of the two central vertices) is the root.

**Theorem 3.** *All trees of diameter five are odd-graceful.*

*Proof.* Let $T$ be a tree of diameter 5 and size $n$. Suppose that $T$ has been drawn according to the previous description. Let $v_{i,j}$ denote the $i$th vertex of level $j$, for $j = 1, 2, 3$; this vertex is placed at the right of $v_{i+1,j}$. Consider the labeling $f$ of the vertices within each level given by recurrence as follows: $f(v) = 0$, where $v$ is the root of $T$, $f(v_{1,1}) = 2n - 2\deg(v) + 1$, $f(v_{1,2}) = 2$, $f(v_{1,3}) = 3$, now the labels are set for the initial vertices of each level and $f(v_{i,j}) = f(v_{i-1,j}) + d(v_{i,j}, v_{i-1,j})$ where $i \geq 2$, $1 \leq j \leq 3$, and $d(v_{i,j}, v_{i-1,j})$ represents the distance between the vertices $v_{i,j}$ and $v_{i-1,j}$.

We claim that $f$ is an odd-graceful labeling of $T$. In fact, let us see that there is no overlapping of labels. On level 0 the label used is 0 and on level 2 all labels are even, being 2 the smallest label used here. On levels 1 and 3 the labels used are odd; on level 1 the labels used are $2n-1, 2n-3, ..., 2n-2\deg(v)+1$, while on level 3 the labels used are $3, 5, ....$ We want to prove that the largest label on level 3 is less than the smallest label on level 1.

Suppose that $k$ is the number of vertices on level 3; thus the weights on level 3 edges are $1, 3, ..., 2k-1$; if $v_{t,2}$ is the last son of $v_{1,1}$ that has sons on level 3, then the weight $2k-1$ must be obtained on the edge $v_{t,2}v_{k,3}$. Since $f(v_{k,3}) = f(v_{t,2})+2k-1 \leq 2\deg(v_{1,1})+2k-3$, we claim that $f(v_{k,3}) < f(v_{1,1})$. In fact, since $\deg(v) + \deg(v_{1,1}) < n - k + 2$, we may conclude that $2\deg(v_{1,1}) + 2k - 3 < 2n - 2\deg(v) - 1$. Hence, the largest label on level 3 is less than the smallest label on level 1, which implies that there is no overlapping of labels.

As a consequence of the fact that labels used in consecutive levels have different parity, each weight obtained is an odd number not exceeding $2n-1$. Suppose that $v_{i+1,j}$ and $v_{i,j}$ have the same father $x$, by definition of $f$, the edges $xv_{i+1,j}$ and $xv_{i,j}$ have consecutive weights. If $v_{i+1,j}$ and $v_{i,j}$ have different father, $x$ and $y$, respectively, then $|f(y) - f(v_{i,j})| = |(f(x)+2) - (f(v_{i+1,j})+4)| = |f(x) - f(v_{i+1,j}) - 2|$. Thus, on level 1 the weights are $2n-1, ..., 2n-2\deg(v)+1$, on level 2, the weights are $2n-2\deg(v)-1, ..., 2k+1$, and on level 3 the weights are $2k-1, ..., 1$.

Therefore, $f$ is an odd-graceful labeling of $T$. □

In Figure 4 we present a scheme of this labeling for a tree of size 23.

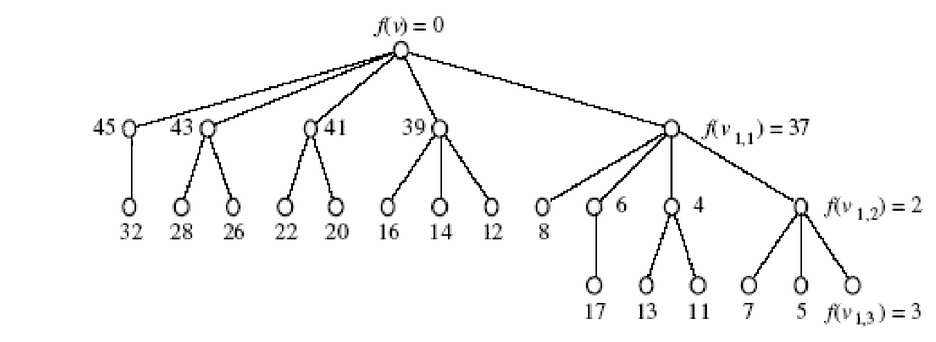


Figure 4: Odd-graceful tree of diameter 5

Similar arguments can be used to find odd-graceful labelings of trees of diameter 6; however we do not have a general labeling scheme for this case. So it is an open problem determining whether trees of diameter 6 are odd-graceful. In Figure 5, we give an example of an odd-graceful labeling for a tree of size 17 and diameter 6.

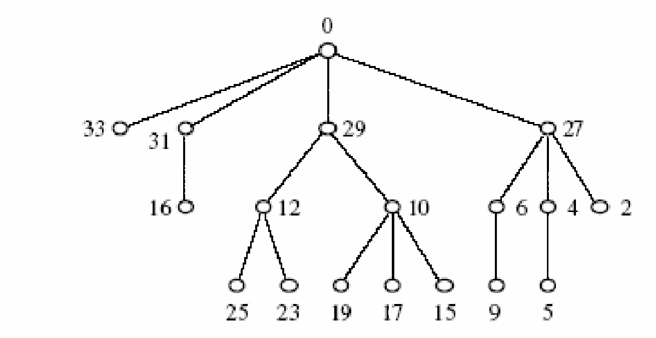


Figure 5: Odd-graceful tree of diameter 6

To conclude this section, we show in Figure 6 an odd-graceful labeling for a special type of tree of diameter 8, namely the star $S(n, 4)$ with $n$ spokes of length 4. The deletion of the vertices in the last row produces the star $S(n, 3)$, a graceful labeling of this tree is obtained by subtracting $2n$ from the labels on the odd-numbered levels.

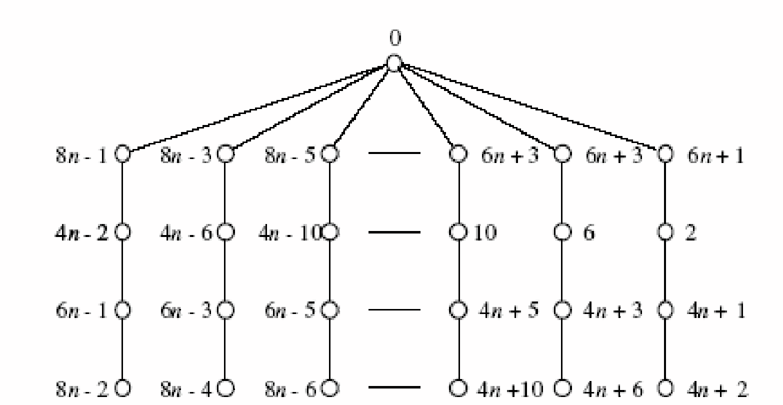


Figure 6: Odd-graceful labeling of the star $S(n, 4)$

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