Automata PhD Qualifying Exam **Name:** ____________________

November 05, 2004 (Friday)

Answer ALL problems.

1. Given a language L, we define a relation $\equiv_L$ on Σ^* as follows:

$$x \equiv_L y \text{ if and only if } \forall \alpha, \beta \in \Sigma^*, \alpha x \beta \in L \Longleftrightarrow \alpha y \beta \in L.$$

   (a) Prove that $\equiv_L$ is an equivalence relation.

   (b) What are the equivalence classes of $\equiv_L$ for $L = \{a^n b^n | n \geq 0\}$?

2. Consider the following definitions:

   Definition: Let Σ be an alphabet and strings $w = a_1 ... a_n \in \Sigma^*$ and $x = b_1 ... b_n \in \Sigma^*$. Then, $alt(w, x) = a_1 b_1 a_2 b_2 ... a_n b_n$.

   Definition: Let L_1 and L_2 be languages over Σ. Then,

$$alt(L_1, L_2) = \{alt(w, x) \mid w \in L_1 \text{ and } x \in L_2 \text{ and } |w| = |x|\}$$

   Prove that if L_1 and L_2 are regular languages over $\Sigma = \{a, b\}$, then $alt(L_1, L_2)$ is also regular.

3. Let $L = \{w \in \{a, b\}^* \mid n_a(w) - n_b(w) \equiv 1 (\text{ mod } 3)\}$. Note that $n_a(w)$ denotes the number of occurrences of a in w. Similarly $n_b(w)$ denotes the number of occurrences of b in w. Let G be the following CFG:

   $S \rightarrow AA \mid aAa \mid a$
   $A \rightarrow SS \mid bSb \mid b$

   where S is the start symbol and $\Sigma = \{a, b\}$.

   (a) Give a DFA that accepts L.

   (b) Convert the DFA to a regular expression using the construction in Kleene's theorem.

   (c) Prove that $L(G) \subseteq L$.

4. Let $L_1 = \{a^{3^n} \mid n \geq 1\}$ and let $L_2 = L_1^*$.

   (a) Is L_1 a regular language? Justify your answer.

   (b) Is L_2 a regular language? Justify your answer.

5. Solve the following two problems.

   (a) Is $L_1 = \{a^n b^n a^m b^m \mid n \geq 1, m \geq 1\}$ a context-free language? Justify your answer.

   (b) Is $L_2 = \{a^n b^m c^n \mid n \geq 1, m > n\}$ a context-free language? Justify your answer.

6. For a language L, define

$$\text{Transpose}(L) = \{w_2 w_1 \mid |w_1| = |w_2| \text{ and } w_1 w_2 \in L\}.$$

True or False: If L is a context-free language, then Transpose(L) is also context-free. Justify your answer.

7. A context-free grammar is *reduced* if, for each variable A,

   (a) S generates a string containing A, and

   (b) A generates a string of terminal symbols.

   A variable A is *cyclic* if A generates (in one or more steps) a string containing A.

   Let G be a reduced CFG with no Λ-productions or unit productions, and assume that $L(G) \neq \emptyset$. Prove that $L(G)$ is infinite if and only if G has a cyclic variable.

8. Let L be the following language:

$$L = \{w \in \{a, b, c\}^* \mid n_a(w) = n_b(w) \quad \text{and} \quad n_b(w) = n_c(w)\}$$

Note that $n_a(w)$ denotes the number of occurrences of a in w. Similarly $n_b(w)$ denotes the number of occurrences of b in w, and $n_c(w)$ denotes the number of occurrences of c in w.

   (a) Describe in English how a Turing machine T could recognize this language.

   (b) Draw the transition diagram for T.

   (c) Using $\vdash$ notation, show the computation of T for input *abacbabcc*.