

## Design and Analysis of Algorithms

Ph.D. Qualifying Exam

Fall 2004

### Notes

- Answer any four. Time = 3 hours. Write your name and the indices of those four questions you want graded on the top right corner of this question paper and attach to your answer sheets. If you fail to identify those four questions, only your first four answers would be graded.
- For each algorithm that you design, (a) describe it stepwise in English, and give pseudo-codes only if clarity demands it, or question needs it, (b) *give a basic justification of its correctness*, and (c) *analyze its time complexity*. For proofs, employ established proof techniques and frameworks.
- The more efficient your (correct) algorithm is, the better your score will be. Also, the more concise and simple your solution is, the better your grade will be. Show your work and convey the ideas you explore, as partial marks will be given.

### Questions

1. Many discrete operations involve the following summation

$$\sum_{i=1}^n i * a[i] =$$

It involves  $n$  multiplications of the index  $i$  and the array elements  $a[i]$ . It is more efficient to transform multiplication into additions without using the logarithm.

Show that the number of multiplications can be reduced to  $n/2+1$ . Then, describe an algorithm to efficiently implement this.

2. A computer takes 1 ns to compute  $\sum_{i=1}^n i$  for  $n=100$ . How long does it take the computer to compute:

a.  $\sum_{i=1}^n (3i^2 - i)$

b.  $\sum_{i=1}^n (i^3 + 125)$

Hint: you may use the following summations:

$$\sum_{i=1}^n i = n(n+1)/2, \quad \sum_{i=1}^n i^2 = n(n+1)(2n+1)/6, \quad \sum_{i=1}^n i^3 = [n^2(n+1)^2]/4.$$

3. In the rectilinear metric the distance between 2 points  $(x,y)$  and  $(x',y')$  equals  $|x-x'|+|y-y'|$ . Give  $O(n)$  algorithm finding the maximum distance between two points out of  $n$  given points in the plane. Hint: Find minimum 45-degree-rotated rectangle  $R$  encompassing all given points.
4. Let  $T$  be a binary search tree  $T=(V,E,k)$  with keys on vertices  $k:V \rightarrow \mathbb{R}$ , and the sum of all keys equal  $S$ . A center  $v$  of  $T$  is a vertex in  $T$  such that each connected component  $C$  of  $T-v$  has sum of keys of all elements at most  $S/2$ . Give an  $O(n)$  algorithm finding a center  $v$  of  $T$ . Hint: Use one of the traversals of  $T$  (post-order/pre-order/in-order)
5. Suppose you are managing a database with records of people living all over the world. Each person's record is uniquely identified by his/her phone number. Note that the length of phone numbers may be different in different countries. For example, United States uses 10-digit phone numbers, Poland uses 9 digit phone numbers, Tunisia uses 8 digit phone numbers, and so on. Some countries even have variable length numbering schemes, where small towns may have 5- or 6-digit numbers, and big cities have 7- or 8-digit numbers.

If you add the length of phone numbers in your database, you get a total of  $N$  digits. Now design an algorithm that can sort the phone numbers in the database in  $O(N)$  time. Prove that your algorithm runs in  $O(N)$  time.

6. Solve the following two problems:
  - a. Given a sequence of  $n$  numbers, design a  $O(n)$  algorithm to find the longest consecutive monotonically non-decreasing sequence. For example, in  $(9, 7, 8, 1, 5, 6, 8, 8, 1, 1, 9, 2, 6)$ , the longest consecutive monotonically non-decreasing sequence is  $(1, 5, 6, 8, 8)$ .
  - b. Given a sequence of  $n$  numbers, design a  $O(n^2)$  algorithm to find the longest monotonically non-decreasing sequence. Note that the "longest monotonically non-decreasing sequence" may not be consecutive. For example, in  $(0, 3, 2, 1, 1, 7, 3, 8, 3, 9)$ , the longest monotonically non-decreasing sequence is  $(0, 1, 1, 7, 8, 9)$ . Hint: You may use dynamic programming