Complexities of π

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Outline

Approximate Value of π
(First 400 Digits After Decimal Point)

$$
\begin{aligned}
\pi = 3.&1415926535897932384626433832795028841971\\
&6939937510582097494459230781640628620899\\
&8628034825342117067982148086513282306647\\
&0938446095505822317253594081284811174502\\
&8410270193852110555964462294895493038196\\
&4428810975665933446128475648233786783165\\
&2712019091456485669234603486104543266482\\
&1339360726024914127372458700660631558817\\
&4881520920962829254091715364367892590360\\
&0113305305488204665213841469519415116094\ldots
\end{aligned}
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- Ratio of the circumference of a circle to its diameter
- Radian angle measure
- Formulas for areas, volumes
- A lot of other really cool (and unexpected!) places

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Power Series

Power Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

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Convergence of a Power Series

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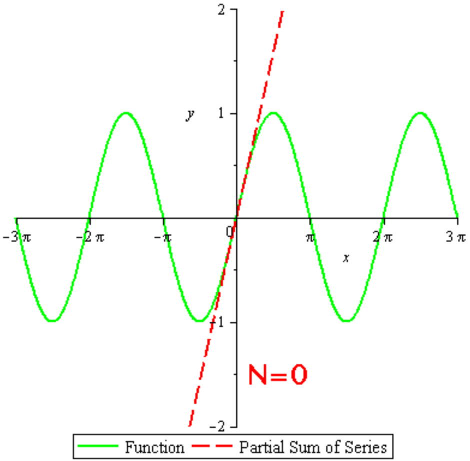

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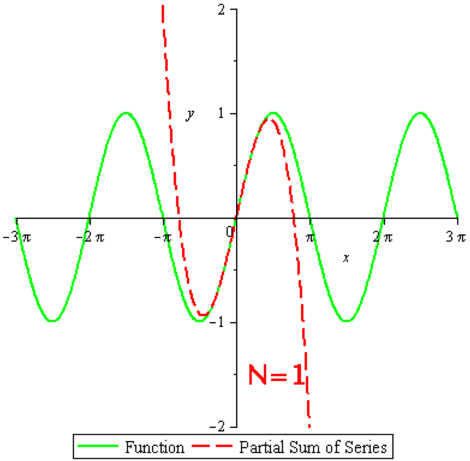

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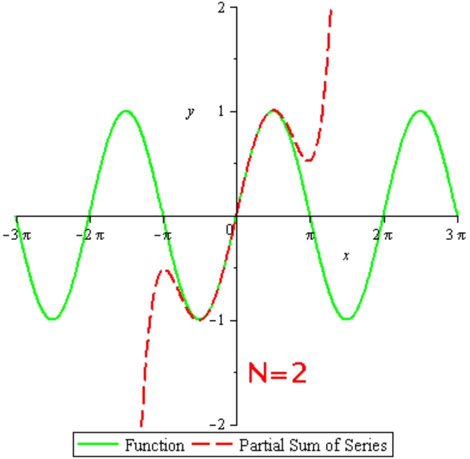

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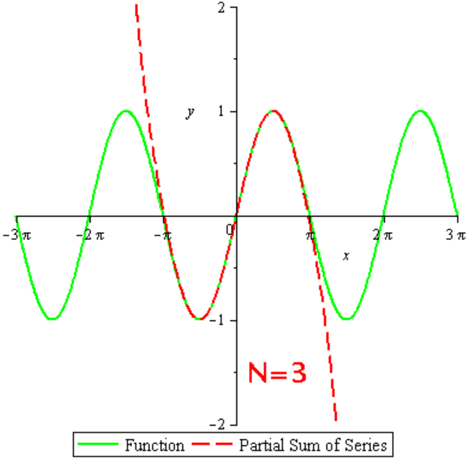

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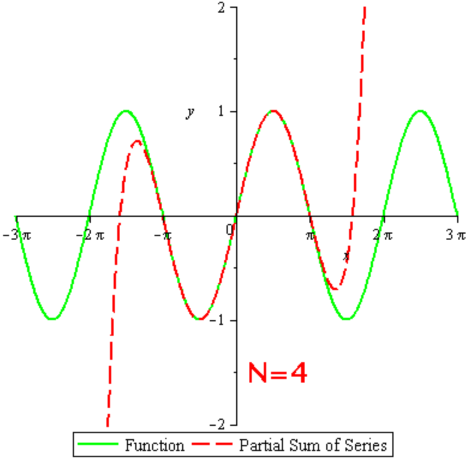

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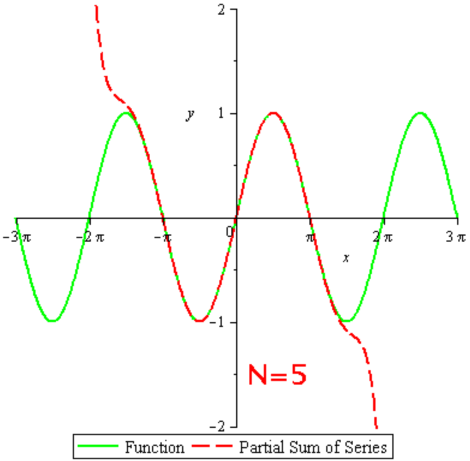

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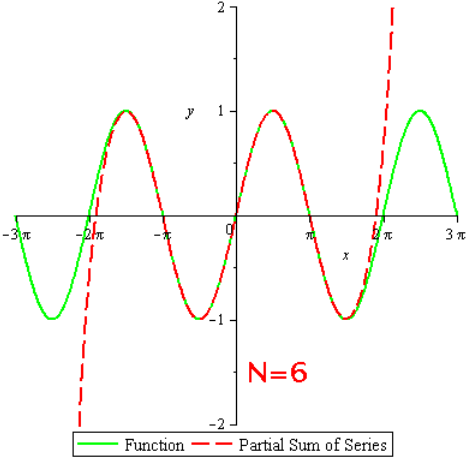

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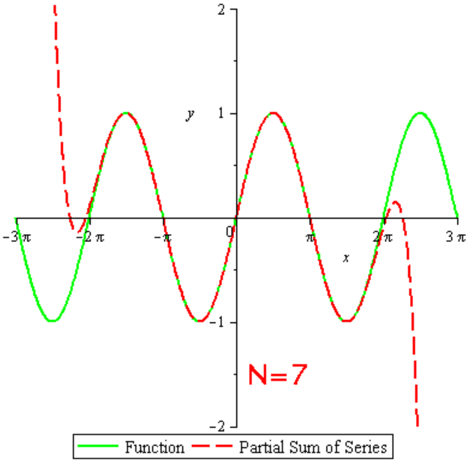

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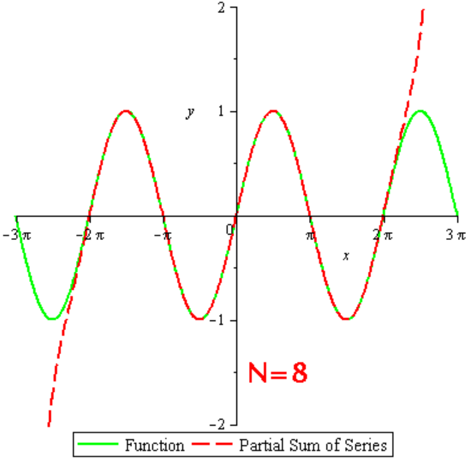

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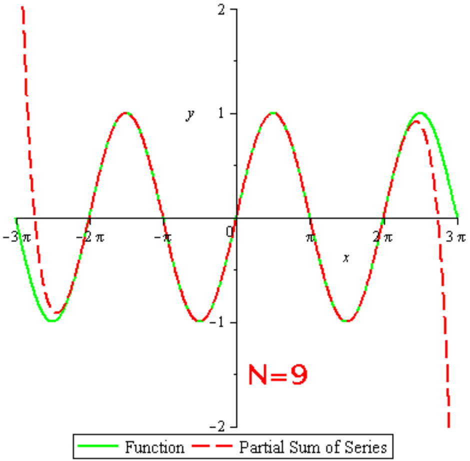

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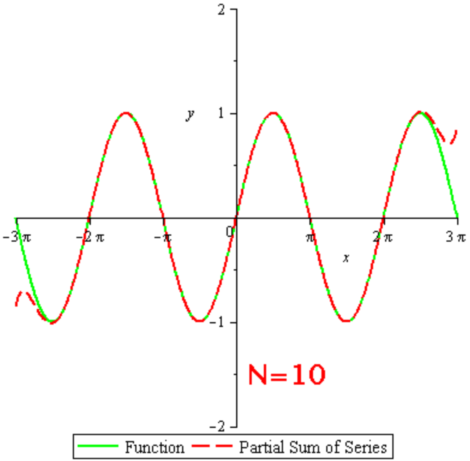

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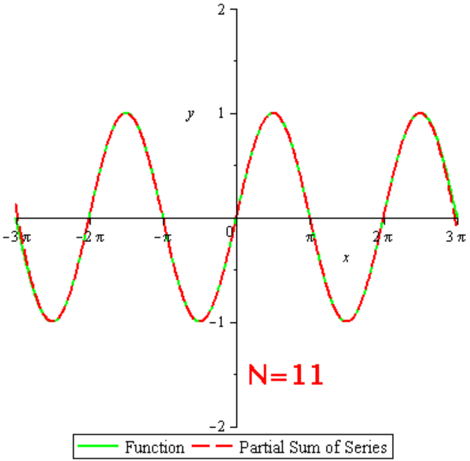

Deriving Euler's Formula

$$
\begin{aligned}
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}
$$

The Punchline

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This result is valid for all θ, so let $\theta = \pi$.

$$e^{i\pi} + 1 = 0$$

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A Surprising Integral

Consider the improper integral

$$I = \int_0^\infty \frac{dx}{1+x^2}.$$

We know from calculus that the value of the integral is

$$I = \lim_{T\to\infty} \int_0^T \frac{dx}{1+x^2} = \lim_{T\to\infty} \left(\arctan(x) \Big|_0^T \right) = \frac{\pi}{2}$$

But what if we think about the integral using complex variables …

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But what if we think about the integral using complex variables ...

The Surprising Integral Revisited

Note: $x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i)$.

So we can rewrite the antiderivative as follows:

$$\int \frac{dx}{1 + x^2} = \frac{1}{2i} \int \left(\frac{1}{x - i} - \frac{1}{x + i} \right) dx = \ln \left(\frac{x - i}{x + i} \right) + C$$

Therefore,

$$\int_0^\infty \frac{dx}{1 + x^2} = \frac{1}{2i} \ln \left(\frac{x - i}{x + i} \right) \Big|_0^\infty$$

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we must conclude that

$$\frac{\pi}{2} = -\frac{1}{2i} \ln\left(\frac{-i}{i}\right) = \frac{i}{2}\left(\ln(-i) - \ln(i)\right)$$

$$= \frac{i}{2}\left(\ln\left(\frac{1}{i} - \ln(i)\right)\right) = -i\ln(i).$$

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Summary

- The study of complex variables leads to many fascinating results involving π
- Many of these results are also inextricably tied to the constants "i" and "e"
- So get a book aboout complex variable theory, or take a class in it!
- Next year we should call it "πie Day"

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Questions?