A Brief Introduction to Multi-Scale Asymptotics: The Amplitude Equation Method

David B. Williams, Ph.D.
Assistant Professor of Mathematics

September 27, 2005

Abstract

A number of specialized techniques have evolved to analyze multi-scale problems asymptotically. Having arisen from differing needs, these techniques all appear to be fundamentally different in character. Synthesizing the best features of classical methods, an amplitude equation approach is derived and shown to give identical or superior results. For periodic solutions, it is shown that one can employ the amplitude equations to determine the asymptotic solution valid to a given order of error on arbitrarily long timescales. The amplitude equation methodology may be extended to encompass oscillators with slowly-varying frequencies and certain systems with both fast and slow dynamics.

Background

If for two given functions

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{g(\epsilon)} = A,$$

for some nonzero constant A,

Background

If for two given functions

$$\lim_{\epsilon \longrightarrow 0} \frac{f(\epsilon)}{g(\epsilon)} = A,$$

for some nonzero constant A, we write $f(\epsilon) = \mathcal{O}\left(g(\epsilon)\right)$ for $\epsilon \ll 1$.

Background

If for two given functions

$$\lim_{\epsilon\to 0}\frac{f(\epsilon)}{g(\epsilon)}=A,$$

for some nonzero constant A, we write $f(\epsilon)=\mathcal{O}\left(g(\epsilon)\right)$ for $\epsilon \ll 1$.

For example,

$$\sin(\epsilon)=\mathcal{O}(\epsilon), \qquad e^{-\epsilon}=\mathcal{O}(1), \qquad \cos(\epsilon)-1=\mathcal{O}\left(\epsilon^{2}\right)$$

as $\epsilon \longrightarrow 0$.

Background

If for two given functions

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{g(\epsilon)} = A,$$

for some nonzero constant A, we write $f(\epsilon) = \mathcal{O}\left(g(\epsilon)\right)$ for $\epsilon \ll 1$.

For example,

$$\sin(\epsilon) = \mathcal{O}(\epsilon), \qquad e^{-\epsilon} = \mathcal{O}(1), \qquad \cos(\epsilon) - 1 = \mathcal{O}\left(\epsilon^2\right)$$

as $\epsilon \to 0$. (Which follow from the respective Maclaurin series.)

Background

If for two given functions

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{g(\epsilon)} = A,$$

for some nonzero constant A, we write $f(\epsilon) = \mathcal{O}\left(g(\epsilon)\right)$ for $\epsilon \ll 1$.

For example,

$$\sin(\epsilon) = \mathcal{O}(\epsilon), \qquad e^{-\epsilon} = \mathcal{O}(1), \qquad \cos(\epsilon) - 1 = \mathcal{O}\left(\epsilon^2\right)$$

as $\epsilon \to 0$. (Which follow from the respective Maclaurin series.)

If $A = 0$, then we write $f(\epsilon) = o\left(g(\epsilon)\right)$.

Background

If for two given functions

$$\lim_{\epsilon \to 0} \frac{f(\epsilon)}{g(\epsilon)} = A,$$

for some nonzero constant A, we write $f(\epsilon) = \mathcal{O}(g(\epsilon))$ for $\epsilon \ll 1$.

For example,

$$\sin(\epsilon) = \mathcal{O}(\epsilon), \qquad e^{-\epsilon} = \mathcal{O}(1), \qquad \cos(\epsilon) - 1 = \mathcal{O}\left(\epsilon^2\right)$$

as $\epsilon \to 0$. (Which follow from the respective Maclaurin series.)

If $A = 0$, then we write $f(\epsilon) = o(g(\epsilon))$.

A sequence of functions $\{f_k(\epsilon)\}_{k=0}^{\infty}$ such that $f_{k+1}(\epsilon) = o(f_k(\epsilon))$ is called a *gauge sequence*.

Divergent $\neq$ Bad

For all $x > 0$, the exponential integral function has the convergent series expansion

$$\mathrm{Ei}(x) \equiv \int_x^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \log(x) - \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!}.$$

Divergent $\neq$ Bad

For all $x > 0$, the exponential integral function has the convergent series expansion

$$\text{Ei}(x) \equiv \int_x^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \log(x) - \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!}.$$

Alternatively, repeated integrations by parts yield the *divergent* series expansion

$$\text{Ei}(x) = e^{-x} \sum_{m=0}^{N} (-1)^{m-1} \frac{(m-1)!}{x^m} + R_N,$$

Divergent $\neq$ Bad

For all $x > 0$, the exponential integral function has the convergent series expansion

$$\text{Ei}(x) \equiv \int_x^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \log(x) - \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!}.$$

Alternatively, repeated integrations by parts yield the *divergent* series expansion

$$\text{Ei}(x) = e^{-x} \sum_{m=0}^{N} (-1)^{m-1} \frac{(m-1)!}{x^m} + R_N,$$

where the remainder is

$$R_N = (-1)^N N! \int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt.$$

Divergent $\neq$ Bad

For all $x > 0$, the exponential integral function has the convergent series expansion

$$\mathrm{Ei}(x) \equiv \int_x^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \log(x) - \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!}.$$

Alternatively, repeated integrations by parts yield the *divergent* series expansion

$$\mathrm{Ei}(x) = e^{-x} \sum_{m=0}^{N} (-1)^{m-1} \frac{(m-1)!}{x^m} + R_N,$$

where the remainder is

$$R_N = (-1)^N N! \int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt.$$

It can be shown that

$$|R_N| = \mathcal{O}\left(e^{-x} x^{-N-1}\right), \quad \text{as } x \to \infty.$$

Divergent $\neq$ Bad

Applying the ratio test to the divergent series, we find

$$\lim_{N\to\infty}\left|\frac{(N+1)\text{ term}}{N\text{ term}}\right| = \lim_{N\to\infty}\frac{N}{x} = \infty, \qquad \text{for fixed } x.$$

Divergent $\neq$ Bad

Applying the ratio test to the divergent series, we find

$$\lim_{N\to\infty}\left|\frac{(N+1)\text{ term}}{N\text{ term}}\right| = \lim_{N\to\infty}\frac{N}{x} = \infty, \qquad \text{for fixed } x.$$

But since $|R_N| \longrightarrow 0$ as $x \longrightarrow \infty$, it is still possible to get a good approximation to Ei(x) using the divergent series.

Divergent $\neq$ Bad

Applying the ratio test to the divergent series, we find

$$\lim_{N\to\infty}\left|\frac{(N+1)\text{ term}}{N\text{ term}}\right| = \lim_{N\to\infty}\frac{N}{x} = \infty, \qquad \text{for fixed } x.$$

But since $|R_N| \longrightarrow 0$ as $x \longrightarrow \infty$, it is still possible to get a good approximation to $\mathrm{Ei}(x)$ using the divergent series. In fact, for any given x, there is a “minimum” error achieved by taking certain number of terms.

Divergent $\neq$ Bad

Applying the ratio test to the divergent series, we find

$$\lim_{N\to\infty}\left|\frac{(N+1)\text{ term}}{N\text{ term}}\right| = \lim_{N\to\infty}\frac{N}{x} = \infty, \qquad \text{for fixed } x.$$

But since $|R_N| \to 0$ as $x \to \infty$, it is still possible to get a good approximation to Ei(x) using the divergent series. In fact, for any given x, there is a “minimum” error achieved by taking certain number of terms.

	Ei(5)	Ei(10)	Ei(20)
Exact	0.0011482956	$0.4156968930 * 10^{-5}$	$0.9835525291 * 10^{-10}$
Asymptotic	0.0011729418	$0.4156165263 * 10^{-5}$	$0.9835525173 * 10^{-10}$
terms used	5	10	20
correct digits	2	4	7
Convergent	0.0011747398	$0.4157216223 * 10^{-5}$	$0.9835525173 * 10^{-10}$
terms used	17	39	80
correct digits	2	4	7

Switching Gears: A Prototype Example

Consider the solution of the initial value problem

$$\ddot{y} + y + \epsilon(\epsilon - 2)y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Switching Gears: A Prototype Example

Consider the solution of the initial value problem

$$\ddot{y} + y + \epsilon(\epsilon - 2)y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Since this problem is 2^{nd}-order linear with constant coefficients, the solution is (relatively) easy to find.

Switching Gears: A Prototype Example

Consider the solution of the initial value problem

$$\ddot{y} + y + \epsilon(\epsilon - 2)y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Since this problem is 2^{nd}-order linear with constant coefficients, the solution is (relatively) easy to find. In fact, the unique solution is

$$y(t; \epsilon) = \cos(t - \epsilon t).$$

Switching Gears: A Prototype Example

Consider the solution of the initial value problem

$$\ddot{y} + y + \epsilon(\epsilon - 2)y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Since this problem is 2^{nd}-order linear with constant coefficients, the solution is (relatively) easy to find. In fact, the unique solution is

$$y(t;\epsilon) = \cos(t - \epsilon t).$$

The solution depends explicitly on two times: the original time t and a *slow time*, $\sigma \equiv \epsilon t$.

Switching Gears: A Prototype Example

Consider the solution of the initial value problem

$$\ddot{y} + y + \epsilon(\epsilon - 2)y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0.$$

Since this problem is 2^{nd}-order linear with constant coefficients, the solution is (relatively) easy to find. In fact, the unique solution is

$$y(t; \epsilon) = \cos(t - \epsilon t).$$

The solution depends explicitly on two times: the original time t and a *slow time*, $\sigma \equiv \epsilon t$.

The equation above is a member of the larger class of (autonomous) weakly-nonlinear oscillators

$$\ddot{y} + y + \epsilon f\,(y, \dot{y}, \epsilon) = 0.$$

Prototype Example: Take 2

Pretend we don’t know the exact solution.

Prototype Example: Take 2

Pretend we don’t know the exact solution. Instead seek a *regular perturbation solution* in the form of the series

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots .$$

Substitute the series into the IVP and collect like powers of ϵ.

Prototype Example: Take 2

Pretend we don't know the exact solution. Instead seek a *regular perturbation solution* in the form of the series

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots .$$

Substitute the series into the IVP and collect like powers of ϵ. We find:

$$\begin{aligned} \ddot{y}_0 + y_0 &= 0, \\ \ddot{y}_1 + y_1 &= 2y_0, \\ \ddot{y}_k + y_k &= 2y_{k-1} - y_{k-2}, \quad k \geq 2, \end{aligned}$$

Prototype Example: Take 2

Pretend we don't know the exact solution. Instead seek a *regular perturbation solution* in the form of the series

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots .$$

Substitute the series into the IVP and collect like powers of ϵ. We find:

$$\begin{aligned} \ddot{y}_0 + y_0 &= 0, \\ \ddot{y}_1 + y_1 &= 2y_0, \\ \ddot{y}_k + y_k &= 2y_{k-1} - y_{k-2}, \quad k \geq 2, \end{aligned}$$

subject to initial conditions $y_0(0) = 1$, $\dot{y}_0(0) = 0$ and $y_m(0) = \dot{y}_m(0) = 0$, for $m \geq 1$.

Prototype Example: Take 2

Pretend we don't know the exact solution. Instead seek a *regular perturbation solution* in the form of the series

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots .$$

Substitute the series into the IVP and collect like powers of ϵ. We find:

$$\begin{aligned} \ddot{y}_0 + y_0 &= 0, \\ \ddot{y}_1 + y_1 &= 2y_0, \\ \ddot{y}_k + y_k &= 2y_{k-1} - y_{k-2}, \quad k \geq 2, \end{aligned}$$

subject to initial conditions $y_0(0) = 1$, $\dot{y}_0(0) = 0$ and $y_m(0) = \dot{y}_m(0) = 0$, for $m \geq 1$.

The first four terms of the regular perturbation series are

$$y(t;\epsilon) = \cos(t) + \epsilon t \sin(t) - \frac{\epsilon^2}{2} t^2 \cos(t) - \frac{\epsilon^3}{6} t^3 \sin(t) + \mathcal{O}\left(\epsilon^4 t\right).$$

Prototype Example: Take 2

Pretend we don't know the exact solution. Instead seek a *regular perturbation solution* in the form of the series

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots \,.$$

Substitute the series into the IVP and collect like powers of ϵ. We find:

$$\begin{aligned} \ddot{y}_0 + y_0 &= 0, \\ \ddot{y}_1 + y_1 &= 2y_0, \\ \ddot{y}_k + y_k &= 2y_{k-1} - y_{k-2}, \quad k \geq 2, \end{aligned}$$

subject to initial conditions $y_0(0) = 1$, $\dot{y}_0(0) = 0$ and $y_m(0) = \dot{y}_m(0) = 0$, for $m \geq 1$.

The first four terms of the regular perturbation series are

$$y(t;\epsilon) = \cos(t) + \epsilon t \sin(t) - \frac{\epsilon^2}{2} t^2 \cos(t) - \frac{\epsilon^3}{6} t^3 \sin(t) + \mathcal{O}\left(\epsilon^4 t\right) .$$

Compare this with the Maclaurin expansion of $\cos(t - \epsilon t)$.

Prototype Example: Take 3

The terms of the series

$$y(t;\epsilon) = \cos(t) + \epsilon t\sin(t) - \frac{\epsilon^2}{2}t^2\cos(t) - \frac{\epsilon^3}{6}t^3\sin(t) + \mathcal{O}\left(\epsilon^4 t\right).$$

become “ambiguously ordered” when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

Prototype Example: Take 3

The terms of the series

$$y(t;\epsilon) = \cos(t) + \epsilon t\sin(t) - \frac{\epsilon^2}{2}t^2\cos(t) - \frac{\epsilon^3}{6}t^3\sin(t) + \mathcal{O}\left(\epsilon^4 t\right).$$

become “ambiguously ordered” when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

We say the series is not uniformly valid when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

Prototype Example: Take 3

The terms of the series

$$y(t;\epsilon) = \cos(t) + \epsilon t \sin(t) - \frac{\epsilon^2}{2} t^2 \cos(t) - \frac{\epsilon^3}{6} t^3 \sin(t) + \mathcal{O}\left(\epsilon^4 t\right).$$

become “ambiguously ordered” when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

We say the series is not uniformly valid when $t = \mathcal{O}\left(\epsilon^{-1}\right)$. This is why this problem is *singularly perturbed*.

Prototype Example: Take 3

The terms of the series

$$y(t;\epsilon) = \cos(t) + \epsilon t\sin(t) - \frac{\epsilon^2}{2}t^2\cos(t) - \frac{\epsilon^3}{6}t^3\sin(t) + \mathcal{O}\left(\epsilon^4 t\right).$$

become “ambiguously ordered” when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

We say the series is not uniformly valid when $t = \mathcal{O}\left(\epsilon^{-1}\right)$. This is why this problem is *singularly perturbed*.

To find a uniformly valid solution, we must try something different.

Prototype Example: Take 3

The terms of the series

$$y(t;\epsilon) = \cos(t) + \epsilon t \sin(t) - \frac{\epsilon^2}{2} t^2 \cos(t) - \frac{\epsilon^3}{6} t^3 \sin(t) + \mathcal{O}\left(\epsilon^4 t\right).$$

become “ambiguously ordered” when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

We say the series is not uniformly valid when $t = \mathcal{O}\left(\epsilon^{-1}\right)$. This is why this problem is *singularly perturbed*.

To find a uniformly valid solution, we must try something different. Since the exact solution depends on two distinct timescales, t and $\sigma = \epsilon t$, treat them as independent and solve a PDE instead.

Prototype Example: Take 3

The terms of the series

$$y(t;\epsilon) = \cos(t) + \epsilon t \sin(t) - \frac{\epsilon^2}{2} t^2 \cos(t) - \frac{\epsilon^3}{6} t^3 \sin(t) + \mathcal{O}\left(\epsilon^4 t\right).$$

become “ambiguously ordered” when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

We say the series is not uniformly valid when $t = \mathcal{O}\left(\epsilon^{-1}\right)$. This is why this problem is *singularly perturbed*.

To find a uniformly valid solution, we must try something different. Since the exact solution depends on two distinct timescales, t and $\sigma = \epsilon t$, treat them as independent and solve a PDE instead.

The differential operators are

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \sigma} \quad \text{and} \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial \sigma} + \epsilon^2 \frac{\partial^2}{\partial \sigma^2}.$$

Prototype Example: Take 3

Letting $y(t) = Y(t, \sigma)$, the new PDE is

$$\frac{\partial^2 Y}{\partial t^2} + Y = -2\epsilon \frac{\partial^2 Y}{\partial t \partial \sigma} - \epsilon^2 \frac{\partial^2 Y}{\partial \sigma^2} - 2\epsilon Y - \epsilon^2 Y$$

Prototype Example: Take 3

Letting $y(t) = Y(t, \sigma)$, the new PDE is

$$\frac{\partial^2 Y}{\partial t^2} + Y = -2\epsilon \frac{\partial^2 Y}{\partial t \partial \sigma} - \epsilon^2 \frac{\partial^2 Y}{\partial \sigma^2} - 2\epsilon Y - \epsilon^2 Y$$

Substitute the series

$$Y(t, \sigma) = Y_0(t, \sigma) + \epsilon Y_1(t, \sigma) + \epsilon^2 Y_2(t, \sigma) + \cdots \ .$$

Prototype Example: Take 3

Letting $y(t) = Y(t, \sigma)$, the new PDE is

$$\frac{\partial^2 Y}{\partial t^2} + Y = -2\epsilon \frac{\partial^2 Y}{\partial t \partial \sigma} - \epsilon^2 \frac{\partial^2 Y}{\partial \sigma^2} - 2\epsilon Y - \epsilon^2 Y$$

Substitute the series

$$Y(t, \sigma) = Y_0(t, \sigma) + \epsilon Y_1(t, \sigma) + \epsilon^2 Y_2(t, \sigma) + \cdots .$$

To leading order, we find

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad Y_0(0, 0) = 1, \; \frac{\partial Y_0}{\partial t}(0, 0) = 0.$$

Prototype Example: Take 3

Letting $y(t) = Y(t, \sigma)$, the new PDE is

$$\frac{\partial^2 Y}{\partial t^2} + Y = -2\epsilon \frac{\partial^2 Y}{\partial t \partial \sigma} - \epsilon^2 \frac{\partial^2 Y}{\partial \sigma^2} - 2\epsilon Y - \epsilon^2 Y$$

Substitute the series

$$Y(t, \sigma) = Y_0(t, \sigma) + \epsilon Y_1(t, \sigma) + \epsilon^2 Y_2(t, \sigma) + \cdots \, .$$

To leading order, we find

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad Y_0(0,0) = 1, \; \frac{\partial Y_0}{\partial t}(0,0) = 0.$$

The general solution of this differential equation is

$$Y_0(t, \sigma) = A_0(\sigma) \cos\left(t + \phi_0(\sigma)\right) .$$

Prototype Example: Take 3

Letting $y(t) = Y(t, \sigma)$, the new PDE is

$$\frac{\partial^2 Y}{\partial t^2} + Y = -2\epsilon \frac{\partial^2 Y}{\partial t \partial \sigma} - \epsilon^2 \frac{\partial^2 Y}{\partial \sigma^2} - 2\epsilon Y - \epsilon^2 Y$$

Substitute the series

$$Y(t, \sigma) = Y_0(t, \sigma) + \epsilon Y_1(t, \sigma) + \epsilon^2 Y_2(t, \sigma) + \cdots \ .$$

To leading order, we find

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad Y_0(0, 0) = 1, \ \frac{\partial Y_0}{\partial t}(0, 0) = 0.$$

The general solution of this differential equation is

$$Y_0(t, \sigma) = A_0(\sigma) \cos\left(t + \phi_0(\sigma)\right) .$$

The initial conditions imply only that $A_0(0) = 1$ and $\phi_0(0) = 0$. We must use other means to determine the slow-time evolution of A_0 and ϕ_0.

Prototype Example: Take 3

Letting $y(t) = Y(t,\sigma)$, the new PDE is

$$\frac{\partial^2 Y}{\partial t^2} + Y = -2\epsilon\frac{\partial^2 Y}{\partial t\partial\sigma} - \epsilon^2\frac{\partial^2 Y}{\partial\sigma^2} - 2\epsilon Y - \epsilon^2 Y$$

Substitute the series

$$Y(t,\sigma) = Y_0(t,\sigma) + \epsilon Y_1(t,\sigma) + \epsilon^2 Y_2(t,\sigma) + \cdots \ .$$

To leading order, we find

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad Y_0(0,0) = 1, \ \frac{\partial Y_0}{\partial t}(0,0) = 0.$$

The general solution of this differential equation is

$$Y_0(t,\sigma) = A_0(\sigma)\cos\left(t + \phi_0(\sigma)\right) .$$

The initial conditions imply only that $A_0(0) = 1$ and $\phi_0(0) = 0$. We must use other means to determine the slow-time evolution of A_0 and ϕ_0.

Prototype Example: Take 3

At $\mathcal{O}\left(\epsilon\right)$, we find (after substituting what we discovered about Y_0)

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 2A_0\left(1 + \phi_0'\right)\cos\left(t + \phi_0\right) + 2A_0'\sin\left(t + \phi_0\right)$$

Prototype Example: Take 3

At $\mathcal{O}\left(\epsilon\right)$, we find (after substituting what we discovered about Y_0)

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 2A_0\left(1 + \phi_0'\right)\cos\left(t + \phi_0\right) + 2A_0'\sin\left(t + \phi_0\right)$$

The forcing contains terms proportional to the homogeneous solutions, so the general solution will be resonant...

Prototype Example: Take 3

At $\mathcal{O}(\epsilon)$, we find (after substituting what we discovered about Y_0)

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 2A_0\left(1+\phi_0'\right)\cos\left(t+\phi_0\right) + 2A_0'\sin\left(t+\phi_0\right)$$

The forcing contains terms proportional to the homogeneous solutions, so the general solution will be resonant...unless we choose A_0 and ϕ_0 to eliminate the *secular terms*.

Prototype Example: Take 3

At $\mathcal{O}(\epsilon)$, we find (after substituting what we discovered about Y_0)

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 2A_0\left(1 + \phi_0'\right)\cos\left(t + \phi_0\right) + 2A_0'\sin\left(t + \phi_0\right)$$

The forcing contains terms proportional to the homogeneous solutions, so the general solution will be resonant...unless we choose A_0 and ϕ_0 to eliminate the *secular terms*.

This requires A_0 and ϕ_0 to satisfy the IVPs

$$1 + \phi_0' = 0,\ \phi_0(0) = 0 \quad \text{and} \quad A_0' = 0,\ A_0(0) = 1.$$

Prototype Example: Take 3

At $\mathcal{O}(\epsilon)$, we find (after substituting what we discovered about Y_0)

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 2A_0 \left(1 + \phi_0'\right) \cos\left(t + \phi_0\right) + 2A_0' \sin\left(t + \phi_0\right)$$

The forcing contains terms proportional to the homogeneous solutions, so the general solution will be resonant...unless we choose A_0 and ϕ_0 to eliminate the *secular terms*.

This requires A_0 and ϕ_0 to satisfy the IVPs

$$1 + \phi_0' = 0,\ \ \phi_0(0) = 0 \quad \text{and} \quad A_0' = 0,\ \ A_0(0) = 1.$$

The unique solutions are

$$A_0(\sigma) = 1 \quad \text{and} \quad \phi_0(\sigma) = -\sigma.$$

Prototype Example: Take 3

At $\mathcal{O}(\epsilon)$, we find (after substituting what we discovered about Y_0)

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 2A_0\left(1+\phi_0'\right)\cos\left(t+\phi_0\right) + 2A_0'\sin\left(t+\phi_0\right)$$

The forcing contains terms proportional to the homogeneous solutions, so the general solution will be resonant...unless we choose A_0 and ϕ_0 to eliminate the *secular terms*.

This requires A_0 and ϕ_0 to satisfy the IVPs

$$1+\phi_0' = 0,\ \phi_0(0) = 0 \quad \text{and} \quad A_0' = 0,\ A_0(0) = 1.$$

The unique solutions are

$$A_0(\sigma) = 1 \quad \text{and} \quad \phi_0(\sigma) = -\sigma.$$

Thus

$$Y(t,\sigma) = \cos(t-\sigma) + \mathcal{O}(\epsilon)\,.$$

Prototype Example: Take 3

At $\mathcal{O}(\epsilon)$, we find (after substituting what we discovered about Y_0)

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = 2A_0\left(1+\phi_0'\right)\cos\left(t+\phi_0\right) + 2A_0'\sin\left(t+\phi_0\right)$$

The forcing contains terms proportional to the homogeneous solutions, so the general solution will be resonant...unless we choose A_0 and ϕ_0 to eliminate the *secular terms*.

This requires A_0 and ϕ_0 to satisfy the IVPs

$$1+\phi_0' = 0, \;\; \phi_0(0) = 0 \quad \text{and} \quad A_0' = 0, \;\; A_0(0) = 1.$$

The unique solutions are

$$A_0(\sigma) = 1 \quad \text{and} \quad \phi_0(\sigma) = -\sigma.$$

Thus

$$Y(t,\sigma) = \cos(t-\sigma) + \mathcal{O}(\epsilon)\,.$$

Higher order terms $Y_k(t,\sigma)$, $k \geq 1$, are identically zero.

Summarize

The method of *multiple scales* guarantees that the perturbation solution we find will be uniformly valid on long time intervals, i.e. when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

Summarize

The method of *multiple scales* guarantees that the perturbation solution we find will be uniformly valid on long time intervals, i.e. when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

This is a great improvement over the $t = \mathcal{O}(1)$ interval over which the regular perturbation series was valid.

Summarize

The method of *multiple scales* guarantees that the perturbation solution we find will be uniformly valid on long time intervals, i.e. when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

This is a great improvement over the $t = \mathcal{O}(1)$ interval over which the regular perturbation series was valid.

Multiple scales is a well-established technique with many extra bells and whistles that can be employed to solve more complicated problems.

Summarize

The method of *multiple scales* guarantees that the perturbation solution we find will be uniformly valid on long time intervals, i.e. when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

This is a great improvement over the $t = \mathcal{O}(1)$ interval over which the regular perturbation series was valid.

Multiple scales is a well-established technique with many extra bells and whistles that can be employed to solve more complicated problems.

Although it is well established (even “classical”), it is difficult to automate for even moderately difficult problems.

Summarize

The method of *multiple scales* guarantees that the perturbation solution we find will be uniformly valid on long time intervals, i.e. when $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

This is a great improvement over the $t = \mathcal{O}(1)$ interval over which the regular perturbation series was valid.

Multiple scales is a well-established technique with many extra bells and whistles that can be employed to solve more complicated problems.

Although it is well established (even “classical”), it is difficult to automate for even moderately difficult problems.

One improvement centers around changing the ODE to a first-order system.

Deriving the Amplitude Equations

Consider the general system

$$\frac{d\mathbf{x}}{d\tau} = \epsilon\mathbf{F}(\mathbf{x}, \tau, \sigma, \epsilon), \;\; \mathbf{x}(0) \;\; \text{prescribed.}$$

Deriving the Amplitude Equations

Consider the general system

$$\frac{d\mathbf{x}}{d\tau} = \epsilon\mathbf{F}(\mathbf{x}, \tau, \sigma, \epsilon), \ \ \mathbf{x}(0) \ \text{ prescribed.}$$

Assume that **F** is smooth in **x** and ϵ and 2π-periodic in the *fast time* τ.

Deriving the Amplitude Equations

Consider the general system

$$\frac{d\mathbf{x}}{d\tau} = \epsilon \mathbf{F}(\mathbf{x}, \tau, \sigma, \epsilon), \;\; \mathbf{x}(0) \;\; \text{prescribed.}$$

Assume that **F** is smooth in **x** and ϵ and 2π-periodic in the *fast time* τ.

Also assume that **F** is smooth in the *slow time* σ.

Deriving the Amplitude Equations

Consider the general system

$$\frac{d\mathbf{x}}{d\tau} = \epsilon \mathbf{F}(\mathbf{x}, \tau, \sigma, \epsilon), \quad \mathbf{x}(0) \ \text{prescribed.}$$

Assume that **F** is smooth in **x** and ϵ and 2π-periodic in the *fast time* τ.

Also assume that **F** is smooth in the *slow time* σ. The “extra” dependence on the slow time might, for example, derive from a slowly-varying coefficient in the ODE.

Deriving the Amplitude Equations

Consider the general system

$$\frac{d\mathbf{x}}{d\tau} = \epsilon \mathbf{F}(\mathbf{x}, \tau, \sigma, \epsilon), \;\; \mathbf{x}(0) \;\; \text{prescribed.}$$

Assume that **F** is smooth in **x** and ϵ and 2π-periodic in the *fast time* τ.

Also assume that **F** is smooth in the *slow time* σ. The “extra” dependence on the slow time might, for example, derive from a slowly-varying coefficient in the ODE.

Introduce the near-identity transformation

$$\mathbf{x}_\epsilon(\tau, \sigma) = \mathbf{A}_\epsilon(\sigma) + \epsilon \mathbf{U}(\mathbf{A}_\epsilon(\sigma), \tau, \sigma, \epsilon),$$

Deriving the Amplitude Equations

Consider the general system

$$\frac{d\mathbf{x}}{d\tau} = \epsilon \mathbf{F}(\mathbf{x}, \tau, \sigma, \epsilon), \;\; \mathbf{x}(0) \;\; \text{prescribed.}$$

Assume that **F** is smooth in **x** and ϵ and 2π-periodic in the *fast time* τ.

Also assume that **F** is smooth in the *slow time* σ. The “extra” dependence on the slow time might, for example, derive from a slowly-varying coefficient in the ODE.

Introduce the near-identity transformation

$$\mathbf{x}_\epsilon(\tau, \sigma) = \mathbf{A}_\epsilon(\sigma) + \epsilon \mathbf{U}(\mathbf{A}_\epsilon(\sigma), \tau, \sigma, \epsilon),$$

where the amplitude, $\mathbf{A}_\epsilon$, satisfies some (generally nonlinear) initial value problem

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} = \omega(\sigma)\, \mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon), \;\; \mathbf{A}_\epsilon(0) = \mathbf{x}(0)$$

Deriving the Amplitude Equations

Consider the general system

$$\frac{d\mathbf{x}}{d\tau} = \epsilon \mathbf{F}(\mathbf{x}, \tau, \sigma, \epsilon), \;\; \mathbf{x}(0) \;\; \text{prescribed.}$$

Assume that **F** is smooth in **x** and ϵ and 2π-periodic in the *fast time* τ.

Also assume that **F** is smooth in the *slow time* σ. The “extra” dependence on the slow time might, for example, derive from a slowly-varying coefficient in the ODE.

Introduce the near-identity transformation

$$\mathbf{x}_\epsilon(\tau, \sigma) = \mathbf{A}_\epsilon(\sigma) + \epsilon \mathbf{U}(\mathbf{A}_\epsilon(\sigma), \tau, \sigma, \epsilon),$$

where the amplitude, $\mathbf{A}_\epsilon$, satisfies some (generally nonlinear) initial value problem

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} = \omega(\sigma)\, \mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon), \;\; \mathbf{A}_\epsilon(0) = \mathbf{x}(0)$$

and the correction term **U** is smooth and 2π-periodic in τ.

Deriving the Amplitude Equations

This is clearly a two-time procedure with τ and σ being treated as independent variables.

Deriving the Amplitude Equations

This is clearly a two-time procedure with τ and σ being treated as independent variables.

Note also that the fast time τ is defined by $\frac{d\tau}{dt} = \omega(\sigma)$...more about this later.

Deriving the Amplitude Equations

This is clearly a two-time procedure with τ and σ being treated as independent variables.

Note also that the fast time τ is defined by $\frac{d\tau}{dt} = \omega(\sigma)$...more about this later.

As in multiple scales, the chain rule and the original problem then define the PDE

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon} \frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon \frac{\partial \mathbf{U}}{\partial \sigma} + \omega(\sigma) \frac{\partial \mathbf{U}}{\partial \tau} = \omega(\sigma)\, \mathbf{F}(\mathbf{A}_\epsilon + \epsilon \mathbf{U}, t, \epsilon).$$

Deriving the Amplitude Equations

This is clearly a two-time procedure with τ and σ being treated as independent variables.

Note also that the fast time τ is defined by $\frac{d\tau}{dt} = \omega(\sigma)$...more about this later.

As in multiple scales, the chain rule and the original problem then define the PDE

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon} \frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon \frac{\partial \mathbf{U}}{\partial \sigma} + \omega(\sigma) \frac{\partial \mathbf{U}}{\partial \tau} = \omega(\sigma)\, \mathbf{F}(\mathbf{A}_\epsilon + \epsilon \mathbf{U}, t, \epsilon).$$

We now average over $0 \leq \tau \leq 2\pi$, using the customary notation

$$\langle g \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(s)\, ds.$$

Deriving the Amplitude Equations

This is clearly a two-time procedure with τ and σ being treated as independent variables.

Note also that the fast time τ is defined by $\frac{d\tau}{dt} = \omega(\sigma)$...more about this later.

As in multiple scales, the chain rule and the original problem then define the PDE

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon\frac{\partial \mathbf{U}}{\partial \sigma} + \omega(\sigma)\,\frac{\partial \mathbf{U}}{\partial \tau} = \omega(\sigma)\,\mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, t, \epsilon).$$

We now average over $0 \leq \tau \leq 2\pi$, using the customary notation

$$\langle g \rangle = \frac{1}{2\pi}\int_0^{2\pi} g(s)\,ds.$$

Since **U** is periodic, $\frac{\partial \mathbf{U}}{\partial t}$ must have a zero average. Moreover, $\frac{d\mathbf{A}_\epsilon}{d\sigma}$ will be its own average.

Deriving the Amplitude Equations

This is clearly a two-time procedure with τ and σ being treated as independent variables.

Note also that the fast time τ is defined by $\frac{d\tau}{dt} = \omega(\sigma)$...more about this later.

As in multiple scales, the chain rule and the original problem then define the PDE

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon} \frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon \frac{\partial \mathbf{U}}{\partial \sigma} + \omega(\sigma) \frac{\partial \mathbf{U}}{\partial \tau} = \omega(\sigma)\, \mathbf{F}(\mathbf{A}_\epsilon + \epsilon \mathbf{U}, t, \epsilon).$$

We now average over $0 \leq \tau \leq 2\pi$, using the customary notation

$$\langle g \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(s)\, ds.$$

Since $\mathbf{U}$ is periodic, $\frac{\partial \mathbf{U}}{\partial t}$ must have a zero average. Moreover, $\frac{d\mathbf{A}_\epsilon}{d\sigma}$ will be its own average.

The balance of averages then implies

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} + \epsilon \left\langle \frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon} \right\rangle \frac{d\mathbf{A}_\epsilon}{d\sigma} = \langle \mathbf{F}(\mathbf{A}_\epsilon + \epsilon \mathbf{U}, t, \epsilon) \rangle \, .$$

Deriving the Amplitude Equations

Thus the previously unspecified forcing function **H** in the amplitude equations must satisfy the integral equation

$$\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) = \left\langle \mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, \tau, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma} \right\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, s, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma} \right)\, ds.$$

Deriving the Amplitude Equations

Thus the previously unspecified forcing function **H** in the amplitude equations must satisfy the integral equation

$$\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) = \left\langle \mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, \tau, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma}\right\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, s, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma}\right)\, ds.$$

Likewise, solving the PDE for $\frac{\partial \mathbf{U}}{\partial \tau}$ and integrating, the correction **U** must satisfy the related integral equation

$$\mathbf{U}(\mathbf{A}_\epsilon, \tau, \sigma, \epsilon) = \int_0^{\tau}\left(\mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, s, \sigma, \epsilon) - \left(\mathrm{I} + \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\right)\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma}\right)\, ds.$$

Deriving the Amplitude Equations

Thus the previously unspecified forcing function **H** in the amplitude equations must satisfy the integral equation

$$\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) = \left\langle \mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, \tau, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma} \right\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, s, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon}\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma} \right)\, ds.$$

Likewise, solving the PDE for $\frac{\partial \mathbf{U}}{\partial \tau}$ and integrating, the correction **U** must satisfy the related integral equation

$$\mathbf{U}(\mathbf{A}_\epsilon, \tau, \sigma, \epsilon) = \int_0^{\tau} \left(\mathbf{F}(\mathbf{A}_\epsilon + \epsilon\mathbf{U}, s, \sigma, \epsilon) - \left(\mathrm{I} + \epsilon\frac{\partial \mathbf{U}}{\partial \mathbf{A}_\epsilon} \right) \mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) - \epsilon\frac{\partial \mathbf{U}}{\partial \sigma} \right)\, ds.$$

Note that **U** will automatically be periodic (i.e. nonresonant) since we have constructed the integrand to be periodic with zero average.

Deriving the Amplitude Equations

Generating power series

$$\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) \sim \sum_{j \geq 0} \mathbf{H}_j(\mathbf{A}_\epsilon, \sigma)\, \epsilon^j \quad \text{and} \quad \mathbf{U}(\mathbf{A}_\epsilon, \tau, \sigma, \epsilon) \sim \sum_{j \geq 0} \mathbf{U}_j(\mathbf{A}_\epsilon, \tau, \sigma)\, \epsilon^j$$

to solve the coupled system of integral equations for **H** and **U** is now straightforward using power series in ϵ and the Taylor series for **F**.

Deriving the Amplitude Equations

Generating power series

$$\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) \sim \sum_{j\geq 0} \mathbf{H}_j(\mathbf{A}_\epsilon, \sigma)\, \epsilon^j \quad \text{and} \quad \mathbf{U}(\mathbf{A}_\epsilon, \tau, \sigma, \epsilon) \sim \sum_{j\geq 0} \mathbf{U}_j(\mathbf{A}_\epsilon, \tau, \sigma)\, \epsilon^j$$

to solve the coupled system of integral equations for **H** and **U** is now straightforward using power series in ϵ and the Taylor series for **F**.

Indeed, they can readily be determined using MAPLE, in a manner more transparent than conventional multiscale techniques.

Deriving the Amplitude Equations

Generating power series

$$\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon) \sim \sum_{j\geq 0} \mathbf{H}_j(\mathbf{A}_\epsilon, \sigma)\, \epsilon^j \ \text{ and } \ \mathbf{U}(\mathbf{A}_\epsilon, \tau, \sigma, \epsilon) \sim \sum_{j\geq 0} \mathbf{U}_j(\mathbf{A}_\epsilon, \tau, \sigma)\, \epsilon^j$$

to solve the coupled system of integral equations for **H** and **U** is now straightforward using power series in ϵ and the Taylor series for **F**.

Indeed, they can readily be determined using MAPLE, in a manner more transparent than conventional multiscale techniques.

Note that the limiting problems

$$\frac{d\mathbf{A}_0}{d\sigma} = \omega(\sigma)\, \mathbf{H}_0(\mathbf{A}_0, \sigma) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}(\mathbf{A}_0, s, 0)\, ds, \ \ \mathbf{A}_0(0) = \mathbf{x}(0)$$

and

$$\mathbf{U}(\mathbf{A}_0, \tau, \sigma) = \int_0^t \left(\mathbf{F}(\mathbf{A}_0, s, \sigma, 0) - \mathbf{H}_0(\mathbf{A}_0, \sigma)\right)\, ds$$

could have been anticipated through simple first-order averaging.

Deriving the Amplitude Equations

At $\mathcal{O}\left(\epsilon^{n}\right)$, $n \geq 1$, we find the useful recursion formulas

$$\begin{aligned} \mathbf{H}_n\left(\mathbf{A}_\epsilon, \sigma\right) = {} & \left\langle \mathcal{F}_n\left(\mathbf{A}_\epsilon,\; \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, \sigma\right)\right\rangle - \frac{1}{\omega(\sigma)}\left\langle \frac{\partial \mathbf{U}_{n-1}}{\partial \sigma}\right\rangle \\ & - \sum_{k=0}^{n-1}\left\langle \frac{\partial \mathbf{U}_k}{\partial \mathbf{A}_\epsilon}\right\rangle \mathbf{H}_{n-k-1}\left(\mathbf{A}_\epsilon, \sigma\right), \end{aligned}$$

and

$$\begin{aligned} \mathbf{U}_n\left(\mathbf{A}_\epsilon, \tau, \sigma\right) = \int_0^\tau \Bigg(& \mathcal{F}_n(\mathbf{A}_\epsilon, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, s, \sigma) - \mathbf{H}_n(\mathbf{A}_\epsilon, \sigma) - \frac{1}{\omega(\sigma)}\frac{\partial \mathbf{U}_{n-1}}{\partial \sigma} \\ & - \sum_{k=0}^{n-1}\frac{\partial \mathbf{U}_k}{\partial \mathbf{A}_\epsilon}\mathbf{H}_{n-k-1}\left(\mathbf{A}_\epsilon, \sigma\right)\Bigg)\, ds. \end{aligned}$$

Deriving the Amplitude Equations

At this point, the given problem has been supplanted by the (not originally obvious) IVP

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} = \omega(\sigma)\,\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon), \;\; \mathbf{A}_\epsilon(0) = \mathbf{x}(0),$$

whose solution is guaranteed to exist, at least on some bounded σ interval.

Deriving the Amplitude Equations

At this point, the given problem has been supplanted by the (not originally obvious) IVP

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} = \omega(\sigma)\,\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon),\ \ \mathbf{A}_\epsilon(0) = \mathbf{x}(0),$$

whose solution is guaranteed to exist, at least on some bounded σ interval. The solution may, of course, exist for even longer times.

One might try to solve the amplitude equation numerically.

Deriving the Amplitude Equations

At this point, the given problem has been supplanted by the (not originally obvious) IVP

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} = \omega(\sigma)\,\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon),\ \ \mathbf{A}_\epsilon(0) = \mathbf{x}(0),$$

whose solution is guaranteed to exist, at least on some bounded σ interval. The solution may, of course, exist for even longer times.

One might try to solve the amplitude equation numerically. On the other hand, we might seek a regular perturbation solution

$$\mathbf{A}_\epsilon(\sigma) = \mathbf{A}_0(\sigma) + \epsilon\mathbf{A}_1(\sigma) + \cdots\ .$$

Deriving the Amplitude Equations

At this point, the given problem has been supplanted by the (not originally obvious) IVP

$$\frac{d\mathbf{A}_\epsilon}{d\sigma} = \omega(\sigma)\,\mathbf{H}(\mathbf{A}_\epsilon, \sigma, \epsilon),\ \ \mathbf{A}_\epsilon(0) = \mathbf{x}(0),$$

whose solution is guaranteed to exist, at least on some bounded σ interval. The solution may, of course, exist for even longer times.

One might try to solve the amplitude equation numerically. On the other hand, we might seek a regular perturbation solution

$$\mathbf{A}_\epsilon(\sigma) = \mathbf{A}_0(\sigma) + \epsilon\mathbf{A}_1(\sigma) + \cdots\ .$$

The leading term, $\mathbf{A}_0$, will satisfy the limiting nonlinear IVP for $\epsilon = 0$, while each later $\mathbf{A}_k$ will satisfy a linearized equation

$$\frac{d\mathbf{A}_k}{d\sigma} = \omega(\sigma)\,\frac{\partial\mathbf{H}}{\partial\mathbf{A}_\epsilon}(\mathbf{A}_0, \sigma, 0)\,\mathbf{A}_k + \alpha_{k-1}(\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}, \sigma),\ \ \mathbf{A}_k(0) = 0.$$

Weakly Nonlinear Oscillators

Suppose we start with a weakly-nonlinear oscillator

$$\ddot{y} + y + \epsilon f(y, \dot{y}, \epsilon) = 0.$$

Weakly Nonlinear Oscillators

Suppose we start with a weakly-nonlinear oscillator

$$\ddot{y} + y + \epsilon f(y, \dot{y}, \epsilon) = 0.$$

One way to convert it to a system in standard form is to use polar coordinates

$$y = \rho \cos(\eta + \phi), \quad \text{and} \quad \dot{y} = -\rho \sin(\eta + \phi)$$

Weakly Nonlinear Oscillators

Suppose we start with a weakly-nonlinear oscillator

$$\ddot{y} + y + \epsilon f(y, \dot{y}, \epsilon) = 0.$$

One way to convert it to a system in standard form is to use polar coordinates

$$y = \rho \cos(\eta + \phi), \quad \text{and} \quad \dot{y} = -\rho \sin(\eta + \phi)$$

where we'll take ρ and ϕ to be time-varying and we've introduced the strained coordinate

$$\eta = (1 + \epsilon\omega(\epsilon)) \; t$$

Weakly Nonlinear Oscillators

Suppose we start with a weakly-nonlinear oscillator

$$\ddot{y} + y + \epsilon f(y, \dot{y}, \epsilon) = 0.$$

One way to convert it to a system in standard form is to use polar coordinates

$$y = \rho \cos(\eta + \phi), \quad \text{and} \quad \dot{y} = -\rho \sin(\eta + \phi)$$

where we'll take ρ and ϕ to be time-varying and we've introduced the strained coordinate

$$\eta = (1 + \epsilon\omega(\epsilon))\ t$$

for an unspecified expansion

$$\omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$

Weakly Nonlinear Oscillators

Suppose we start with a weakly-nonlinear oscillator

$$\ddot{y} + y + \epsilon f(y, \dot{y}, \epsilon) = 0.$$

One way to convert it to a system in standard form is to use polar coordinates

$$y = \rho\cos(\eta + \phi), \quad \text{and} \quad \dot{y} = -\rho\sin(\eta + \phi)$$

where we'll take ρ and ϕ to be time-varying and we've introduced the strained coordinate

$$\eta = (1 + \epsilon\omega(\epsilon))\ t$$

for an unspecified expansion

$$\omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$

(remembering the need to do so in the Poincaré-Lindstedt method).

Weakly Nonlinear Oscillators

Here the consistency condition $\frac{d}{dt}y = \dot{y}$ forces us to have

$$\dot{\rho}\cos(\eta+\phi) - \rho\dot{\phi}\sin(\eta+\phi) = \epsilon\omega\rho\sin(\eta+\phi),$$

Weakly Nonlinear Oscillators

Here the consistency condition $\frac{d}{dt}y = \dot{y}$ forces us to have

$$\dot{\rho}\cos(\eta+\phi) - \rho\dot{\phi}\sin(\eta+\phi) = \epsilon\omega\rho\sin(\eta+\phi),$$

while the differential equation also requires

$$\dot{\rho}\sin(\eta+\phi) + \rho\dot{\phi}\cos(\eta+\phi) = -\epsilon f - \epsilon\omega\rho\cos(\eta+\phi).$$

Weakly Nonlinear Oscillators

Here the consistency condition $\frac{d}{dt}y = \dot{y}$ forces us to have

$$\dot{\rho}\cos(\eta+\phi) - \rho\dot{\phi}\sin(\eta+\phi) = \epsilon\omega\rho\sin(\eta+\phi),$$

while the differential equation also requires

$$\dot{\rho}\sin(\eta+\phi) + \rho\dot{\phi}\cos(\eta+\phi) = -\epsilon f - \epsilon\omega\rho\cos(\eta+\phi).$$

Solving the linear system for $\dot{\rho}$ and $\rho\dot{\phi}$ shows that

$$\mathbf{x} = \begin{pmatrix} \rho \\ \phi \end{pmatrix}$$

will satisfy an initial value problem for a system $\dot{\mathbf{x}} = \epsilon\mathbf{F}(\mathbf{x}, \eta, \epsilon)$

Weakly Nonlinear Oscillators

Here the consistency condition $\frac{d}{dt}y = \dot{y}$ forces us to have

$$\dot{\rho}\cos(\eta+\phi) - \rho\dot{\phi}\sin(\eta+\phi) = \epsilon\omega\rho\sin(\eta+\phi),$$

while the differential equation also requires

$$\dot{\rho}\sin(\eta+\phi) + \rho\dot{\phi}\cos(\eta+\phi) = -\epsilon f - \epsilon\omega\rho\cos(\eta+\phi).$$

Solving the linear system for $\dot{\rho}$ and $\rho\dot{\phi}$ shows that

$$\mathbf{x} = \begin{pmatrix} \rho \\ \phi \end{pmatrix}$$

will satisfy an initial value problem for a system $\dot{\mathbf{x}} = \epsilon\mathbf{F}(\mathbf{x},\eta,\epsilon)$ where

$$\mathbf{F}(\mathbf{x},\eta,\epsilon) = \begin{pmatrix} f(\rho\cos(\eta+\phi), -\rho\sin(\eta+\phi), \epsilon)\sin(\eta+\phi) \\ \\ \frac{1}{\rho}f(\rho\cos(\eta+\phi), -\rho\sin(\eta+\phi), \epsilon)\cos(\eta+\phi) - \omega(\epsilon) \end{pmatrix}.$$

Example: van der Pol Equation

For the van der Pol equation

$$\ddot{y} + y + \epsilon \left(y^2 - 1 \right) \dot{y} = 0, \;\; y(0) = 1, \dot{y}(0) = 0,$$

Example: van der Pol Equation

For the van der Pol equation

$$\ddot{y} + y + \epsilon\left(y^2 - 1\right)\dot{y} = 0,\ \ y(0) = 1, \dot{y}(0) = 0,$$

we get a bare expansion

$$\rho = R_\epsilon + \frac{\epsilon R_\epsilon}{32}\left(-8\sin\left(2\left(\eta + \Phi_\epsilon\right)\right) + R_\epsilon^2\sin\left(4\left(\eta + \Phi_\epsilon\right)\right)\right) + \mathcal{O}\left(\epsilon^2\right),$$

$$\phi = \Phi_\epsilon + \frac{\epsilon}{32}\left(8 - 5R_\epsilon^2 - \left(8 - 4R_\epsilon^2\right)\cos\left(2\left(\eta + \Phi_\epsilon\right)\right) + R_\epsilon^2\cos\left(4\left(\eta + \Phi_\epsilon\right)\right)\right) + \mathcal{O}\left(\epsilon^2\right),$$

Example: van der Pol Equation

For the van der Pol equation

$$\ddot{y} + y + \epsilon \left(y^2 - 1 \right) \dot{y} = 0, \;\; y(0) = 1, \dot{y}(0) = 0,$$

we get a bare expansion

$$\rho = R_\epsilon + \frac{\epsilon R_\epsilon}{32} \left(-8 \sin\left(2\left(\eta + \Phi_\epsilon\right)\right) + R_\epsilon^2 \sin\left(4\left(\eta + \Phi_\epsilon\right)\right) \right) + \mathcal{O}\left(\epsilon^2\right),$$

$$\phi = \Phi_\epsilon + \frac{\epsilon}{32} \left(8 - 5R_\epsilon^2 - \left(8 - 4R_\epsilon^2 \right) \cos\left(2\left(\eta + \Phi_\epsilon\right)\right) + R_\epsilon^2 \cos\left(4\left(\eta + \Phi_\epsilon\right)\right) \right) + \mathcal{O}\left(\epsilon^2\right),$$

and ask that R_ϵ and Φ_ϵ satisfy the decoupled amplitude equations

$$\begin{aligned} \frac{dR_\epsilon}{d\sigma} &= \frac{R_\epsilon}{8}\left(4 - R_\epsilon^2\right) + \epsilon^2 \frac{R_\epsilon^3}{12288}\left(-480 + 102R_\epsilon^2 - 37R_\epsilon^4\right) + \mathcal{O}\left(\epsilon^3\right), \\ \frac{d\Phi_\epsilon}{d\sigma} &= -\omega_0 - \epsilon\left(\omega_1 + \frac{1}{8} - \frac{11}{32}R_\epsilon^2 + \frac{21}{256}R_\epsilon^4\right) - \epsilon^2\omega_2 + \mathcal{O}\left(\epsilon^3\right). \end{aligned}$$

Example: van der Pol

We naturally find R_ϵ as a power series in ϵ^2 with a leading term $R_0(\sigma) = 2/\sqrt{1+e^{-\sigma}}$.

Example: van der Pol

We naturally find R_ϵ as a power series in ϵ^2 with a leading term $R_0(\sigma) = 2/\sqrt{1 + e^{-\sigma}}$.
Clearly, a limit cycle is obtained as $\sigma \longrightarrow \infty$.

Example: van der Pol

We naturally find R_ϵ as a power series in ϵ^2 with a leading term $R_0(\sigma) = 2/\sqrt{1+e^{-\sigma}}$. Clearly, a limit cycle is obtained as $\sigma \longrightarrow \infty$.

Moreover, to get a bounded phase,

$$\Phi_\epsilon(\sigma) = \frac{\epsilon}{64}\left(21\left(R_0^2 - 1\right) - 8\ln R_0\right) + \mathcal{O}\left(\epsilon^3\sigma\right),$$

Example: van der Pol

We naturally find R_ϵ as a power series in ϵ^2 with a leading term $R_0(\sigma) = 2/\sqrt{1+e^{-\sigma}}$. Clearly, a limit cycle is obtained as $\sigma \longrightarrow \infty$.

Moreover, to get a bounded phase,

$$\Phi_\epsilon(\sigma) = \frac{\epsilon}{64}\left(21\left(R_0^2 - 1\right) - 8\ln R_0\right) + \mathcal{O}\left(\epsilon^3\sigma\right),$$

we must take

$$\eta = \left(1 - \frac{1}{16}\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right)t.$$

Example: van der Pol

We naturally find R_ϵ as a power series in ϵ^2 with a leading term $R_0(\sigma) = 2/\sqrt{1+e^{-\sigma}}$. Clearly, a limit cycle is obtained as $\sigma \longrightarrow \infty$.

Moreover, to get a bounded phase,

$$\Phi_\epsilon(\sigma) = \frac{\epsilon}{64}\left(21\left(R_0^2 - 1\right) - 8\ln R_0\right) + \mathcal{O}\left(\epsilon^3\sigma\right),$$

we must take

$$\eta = \left(1 - \frac{1}{16}\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right) t.$$

Reexpanding $\cos(\eta + \Phi_\epsilon(\sigma))$ finally provides the approximation

$$y(t,\epsilon) = R_0\cos\eta + \epsilon\frac{R_0}{64}\left[\left(-11 - 7R_0^2 + 8\ln R_0\right)\sin\eta - 2R_0^2\sin 3\eta\right] + \mathcal{O}\left(\epsilon^2\right).$$

Example: van der Pol

We naturally find R_ϵ as a power series in ϵ^2 with a leading term $R_0(\sigma) = 2/\sqrt{1+e^{-\sigma}}$. Clearly, a limit cycle is obtained as $\sigma \longrightarrow \infty$.

Moreover, to get a bounded phase,

$$\Phi_\epsilon(\sigma) = \frac{\epsilon}{64}\left(21\left(R_0^2 - 1\right) - 8\ln R_0\right) + \mathcal{O}\left(\epsilon^3\sigma\right),$$

we must take

$$\eta = \left(1 - \frac{1}{16}\epsilon^2 + \mathcal{O}\left(\epsilon^4\right)\right) t.$$

Reexpanding $\cos(\eta + \Phi_\epsilon(\sigma))$ finally provides the approximation

$$y(t,\epsilon) = R_0\cos\eta + \epsilon\frac{R_0}{64}\left[\left(-11 - 7R_0^2 + 8\ln R_0\right)\sin\eta - 2R_0^2\sin 3\eta\right] + \mathcal{O}\left(\epsilon^2\right).$$

Periodic Solutions

Changing to polar coordinates guarantees that the amplitude equations will have the form

$$\frac{dR_\epsilon}{d\sigma} = \beta_1(R_\epsilon) + \epsilon h_1(R_\epsilon) + \epsilon^2 h_2(R_\epsilon) + \mathcal{O}\left(\epsilon^3\right),$$

$$\frac{d\Phi_\epsilon}{d\sigma} = -\omega_0 + \frac{\alpha_1(R_\epsilon)}{R_\epsilon} + \epsilon\left(-\omega_1 + k_1(R_\epsilon)\right) + \epsilon^2\left(-\omega_3 + k_2(R_\epsilon)\right) + \mathcal{O}\left(\epsilon^3\right),$$

Periodic Solutions

Changing to polar coordinates guarantees that the amplitude equations will have the form

$$\frac{dR_\epsilon}{d\sigma} = \beta_1(R_\epsilon) + \epsilon h_1(R_\epsilon) + \epsilon^2 h_2(R_\epsilon) + \mathcal{O}\left(\epsilon^3\right) ,$$

$$\frac{d\Phi_\epsilon}{d\sigma} = -\omega_0 + \frac{\alpha_1(R_\epsilon)}{R_\epsilon} + \epsilon\left(-\omega_1 + k_1(R_\epsilon)\right) + \epsilon^2\left(-\omega_3 + k_2(R_\epsilon)\right) + \mathcal{O}\left(\epsilon^3\right) ,$$

Considered in these coordinates, a limit cycle will have a constant (but generally ϵ-dependent) radius, $R_\epsilon(\infty)$.

Periodic Solutions

Changing to polar coordinates guarantees that the amplitude equations will have the form

$$\frac{dR_\epsilon}{d\sigma} = \beta_1(R_\epsilon) + \epsilon h_1(R_\epsilon) + \epsilon^2 h_2(R_\epsilon) + \mathcal{O}\left(\epsilon^3\right),$$

$$\frac{d\Phi_\epsilon}{d\sigma} = -\omega_0 + \frac{\alpha_1(R_\epsilon)}{R_\epsilon} + \epsilon\left(-\omega_1 + k_1(R_\epsilon)\right) + \epsilon^2\left(-\omega_3 + k_2(R_\epsilon)\right) + \mathcal{O}\left(\epsilon^3\right),$$

Considered in these coordinates, a limit cycle will have a constant (but generally ϵ-dependent) radius, $R_\epsilon(\infty)$.

We may solve for this rest point by substituting

$$R_\epsilon = R_\epsilon(\infty) = R_0(\infty) + \epsilon R_1(\infty) + \epsilon^2 R_2(\infty) + \mathcal{O}\left(\epsilon^3\right)$$

into the steady-state.

Periodic Solutions

To $\mathcal{O}\left(\epsilon^3\right)$, the periodic solution for the van der Pol equation is found to be

$$y(t) = 2\cos\eta + \epsilon\left(\frac{3}{4}\sin\eta - \frac{1}{4}\sin 3\eta\right) - \epsilon^2\left(\frac{1}{8}\cos\eta - \frac{3}{16}\cos 3\eta + \frac{5}{96}\cos 5\eta\right)$$

$$- \epsilon^3\left(\frac{7}{256}\sin\eta - \frac{21}{256}\sin 3\eta + \frac{35}{576}\sin 5\eta - \frac{7}{576}\sin 7\eta\right) + \mathcal{O}\left(\epsilon^4\right).$$

Periodic Solutions

To $\mathcal{O}\left(\epsilon^3\right)$, the periodic solution for the van der Pol equation is found to be

$$y(t) = 2\cos\eta + \epsilon\left(\frac{3}{4}\sin\eta - \frac{1}{4}\sin 3\eta\right) - \epsilon^2\left(\frac{1}{8}\cos\eta - \frac{3}{16}\cos 3\eta + \frac{5}{96}\cos 5\eta\right)$$

$$- \epsilon^3\left(\frac{7}{256}\sin\eta - \frac{21}{256}\sin 3\eta + \frac{35}{576}\sin 5\eta - \frac{7}{576}\sin 7\eta\right) + \mathcal{O}\left(\epsilon^4\right).$$

Using the $\mathcal{O}\left(\epsilon^8\right)$ amplitude equations, the corresponding radius is

$$R_\epsilon(\infty) = 2 + \frac{1}{96}\epsilon^2 - \frac{1033}{552960}\epsilon^4 + \frac{1019689}{557738368000}\epsilon^6 + \frac{9835512276689}{157315969843200000}\epsilon^8 + \mathcal{O}\left(\epsilon^{10}\right)$$

Periodic Solutions

To $\mathcal{O}\left(\epsilon^3\right)$, the periodic solution for the van der Pol equation is found to be

$$y(t) = 2\cos\eta + \epsilon\left(\frac{3}{4}\sin\eta - \frac{1}{4}\sin 3\eta\right) - \epsilon^2\left(\frac{1}{8}\cos\eta - \frac{3}{16}\cos 3\eta + \frac{5}{96}\cos 5\eta\right)$$

$$-\epsilon^3\left(\frac{7}{256}\sin\eta - \frac{21}{256}\sin 3\eta + \frac{35}{576}\sin 5\eta - \frac{7}{576}\sin 7\eta\right) + \mathcal{O}\left(\epsilon^4\right).$$

Using the $\mathcal{O}\left(\epsilon^8\right)$ amplitude equations, the corresponding radius is

$$R_\epsilon(\infty) = 2 + \frac{1}{96}\epsilon^2 - \frac{1033}{552960}\epsilon^4 + \frac{1019689}{557383680000}\epsilon^6 + \frac{9835512276689}{1573159698432000000}\epsilon^8 + \mathcal{O}\left(\epsilon^{10}\right)$$

while the strained coordinate is

$$\eta = \left(1 - \frac{1}{16}\epsilon^2 + \frac{17}{3072}\epsilon^4 + \frac{35}{884736}\epsilon^6 - \frac{678899}{5096079360}\epsilon^8 + \mathcal{O}\left(\epsilon^{10}\right)\right)t.$$

Periodic Solutions

To $\mathcal{O}\left(\epsilon^3\right)$, the periodic solution for the van der Pol equation is found to be

$$y(t) = 2\cos\eta + \epsilon\left(\frac{3}{4}\sin\eta - \frac{1}{4}\sin 3\eta\right) - \epsilon^2\left(\frac{1}{8}\cos\eta - \frac{3}{16}\cos 3\eta + \frac{5}{96}\cos 5\eta\right)$$

$$-\epsilon^3\left(\frac{7}{256}\sin\eta - \frac{21}{256}\sin 3\eta + \frac{35}{576}\sin 5\eta - \frac{7}{576}\sin 7\eta\right) + \mathcal{O}\left(\epsilon^4\right).$$

Using the $\mathcal{O}\left(\epsilon^8\right)$ amplitude equations, the corresponding radius is

$$R_\epsilon(\infty) = 2 + \frac{1}{96}\epsilon^2 - \frac{1033}{552960}\epsilon^4 + \frac{1019689}{55738368000}\epsilon^6 + \frac{9835512276689}{157315969843200000}\epsilon^8 + \mathcal{O}\left(\epsilon^{10}\right)$$

while the strained coordinate is

$$\eta = \left(1 - \frac{1}{16}\epsilon^2 + \frac{17}{3072}\epsilon^4 + \frac{35}{884736}\epsilon^6 - \frac{678899}{5096079360}\epsilon^8 + \mathcal{O}\left(\epsilon^{10}\right)\right)t.$$

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

We have applied it to a wide variety of problems and found the results to be comparable or superior to those given by other methods.

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

We have applied it to a wide variety of problems and found the results to be comparable or superior to those given by other methods.

Future Work may include for example:

- Continue to expand the classes of problems for which the amplitude equation method will work

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

We have applied it to a wide variety of problems and found the results to be comparable or superior to those given by other methods.

Future Work may include for example:

- Continue to expand the classes of problems for which the amplitude equation method will work
  - ⋆ Layer-type problems, matched asymptotics

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

We have applied it to a wide variety of problems and found the results to be comparable or superior to those given by other methods.

Future Work may include for example:

- Continue to expand the classes of problems for which the amplitude equation method will work
  - ⋆ Layer-type problems, matched asymptotics
  - ⋆ PDEs

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

We have applied it to a wide variety of problems and found the results to be comparable or superior to those given by other methods.

Future Work may include for example:

- Continue to expand the classes of problems for which the amplitude equation method will work
  - ⋆ Layer-type problems, matched asymptotics
  - ⋆ PDEs
- Apply method to solve problems in application fields
  - ⋆ Oscillators in neuroscience

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

We have applied it to a wide variety of problems and found the results to be comparable or superior to those given by other methods.

Future Work may include for example:

- Continue to expand the classes of problems for which the amplitude equation method will work
  - ⋆ Layer-type problems, matched asymptotics
  - ⋆ PDEs
- Apply method to solve problems in application fields
  - ⋆ Oscillators in neuroscience
  - ⋆ Other biological or chemical processes

Summary and Future Work

The amplitude equation method is a versatile and facile perturbation technique, amenable to automation with a computer algebra system.

We have applied it to a wide variety of problems and found the results to be comparable or superior to those given by other methods.

Future Work may include for example:

- Continue to expand the classes of problems for which the amplitude equation method will work
  - Layer-type problems, matched asymptotics
  - PDEs
- Apply method to solve problems in application fields
  - Oscillators in neuroscience
  - Other biological or chemical processes