

## Section 6.5: Solving Trigonometric Equations

### Key point:

- Know how to solve trig equations.

To **solve** a trig equation means to find the values of the variable for which the trig equation is true. We must isolate the variable!

Sometimes, this means finding the angles that lie in some specified interval, usually  $[0^\circ, 360^\circ)$  or  $[0, 2\pi)$ . Sometimes this means finding **all** solutions using the idea of coterminal angles.

Of course, since solving a trig equation requires using inverse trig functions, we very often must use the ideas of reference angles as well!

EXAMPLE 1. Solve  $2 \sin x = -1$  in  $[0^\circ, 360^\circ)$ .

**Solution:** Here, we are given a specific solution interval; that is, the answers must be angles  $x$  with  $0^\circ \leq x < 360^\circ$ .

Dividing both sides of the original equation by 2 gives

$$\sin x = -\frac{1}{2}$$

and applying the arcsine function to both sides gives

$$\begin{aligned}\sin^{-1}(\sin x) &= \sin^{-1}\left(-\frac{1}{2}\right) \\ x &= -30^\circ.\end{aligned}$$

Now,  $-30^\circ$  is not in  $[0^\circ, 360^\circ)$ . However, we know sine is  $-$  in QIII and QIV, and the angles  $210^\circ$  and  $330^\circ$  are angles in  $[0^\circ, 360^\circ)$  that have the same reference angle as the  $x$  that we found.

Therefore, the solutions are  $\{210^\circ, 330^\circ\}$ .

EXAMPLE 2. Solve  $4 \cos^2 x = 1$  in  $[0, 2\pi)$ .

**Solution:** Here, we are given a specific solution interval; that is, the answers must be angles  $x$  with  $0 \leq x < 2\pi$ . If it is easier for you to “think” in degrees, then use degrees—just remember to convert your final answers to radians!

First, divide both sides by 4 to get

$$\cos^2 x = \frac{1}{4}$$

and then take the square root of both sides to get

$$\cos x = \pm \frac{1}{2}.$$

This means that

$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -\frac{1}{2}.$$

The first of these equations says that cosine is +, which we know is true in QI and QIV. Applying the arccosine function to both sides of  $\cos x = 1/2$  gives

$$\begin{aligned} x &= \cos^{-1} \frac{1}{2} \\ &= \frac{\pi}{3}, \end{aligned}$$

which is the QI solution. The angle in QIV with the same reference angle as this  $x$  is  $\frac{5\pi}{3}$ . Therefore, the solutions to the first equation are  $\left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\}$ .

The second of these equations says that cosine is  $-$ , which we know is true in QII and QIII. Applying the arccosine function to both sides of  $\cos x = -1/2$  gives

$$\begin{aligned} x &= \cos^{-1} \left( -\frac{1}{2} \right) \\ &= \frac{2\pi}{3}, \end{aligned}$$

which is the QII solution. The angle in QIII with the same reference angle as this  $x$  is  $\frac{4\pi}{3}$ . Therefore, the solutions to the second equation are  $\left\{ \frac{2\pi}{3}, \frac{4\pi}{3} \right\}$ .

Thus, we have 4 solutions to the original equation:

$$\left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}.$$

**EXAMPLE 3.** Solve  $2 \sin^2 u = 1 - \sin u$ .

**Solution:** In this example, we are not given any specific solution interval, so we should think “there are infinitely many solutions”. However, these solutions can all be expressed in terms of coterminal angles! We may use radians or degrees, so for convenience, let’s choose degrees.

Moving everything over to the left-hand side gives

$$2 \sin^2 u + \sin u - 1 = 0,$$

which we should recognize is *quadratic in form*. Factoring gives

$$(2 \sin u - 1)(\sin u + 1) = 0,$$

and setting each factor equal to 0 will provide the solutions.

First, we solve

$$2 \sin u - 1 = 0 \implies \sin u = \frac{1}{2}.$$

This says that sine is +, which is true in QI and QII. Applying the arcsine function to both sides gives

$$\begin{aligned} u &= \sin^{-1} \frac{1}{2} \\ &= 30^\circ, \end{aligned}$$

which is the QI solution. The angle in QII with the same reference angle as this  $u$  is  $150^\circ$ . Therefore, the solutions from the first factor are

$$\{30^\circ + k \cdot 360^\circ, 150^\circ + k \cdot 360^\circ\},$$

where  $k$  is any integer. We have expressed the solutions in terms of the angles that are *coterminal* with  $30^\circ$  and  $150^\circ$ .

Next, we solve

$$\sin u + 1 = 0 \implies \sin u = -1.$$

This gives

$$\begin{aligned} u &= \sin^{-1}(-1) \\ &= -90^\circ, \end{aligned}$$

which is a *quadrantal* angle. Since quadrantal angles have no reference angle, the solutions from the second factor are

$$\{-90^\circ + k \cdot 360^\circ\},$$

where  $k$  is any integer. Of course, since  $-90^\circ$  is coterminal with  $270^\circ$ , we could also express these solutions as

$$\{270^\circ + k \cdot 360^\circ\},$$

where  $k$  is any integer.

Therefore, we have infinitely many solutions to the original equation:

$$\{-90^\circ + k \cdot 360^\circ, 30^\circ + k \cdot 360^\circ, 150^\circ + k \cdot 360^\circ\},$$

where  $k$  is any integer.

EXAMPLE 4. Solve

$$\frac{1}{4} \tan \beta + 1 = 0.6237$$

in  $[0^\circ, 360^\circ)$ . Round the angles to the nearest tenth of a degree.

**Solution:** Subtracting 1 from both sides gives

$$\frac{1}{4} \tan \beta = -0.3763.$$

Multiplying both sides by 4 gives

$$\tan \beta = -1.5052$$

and taking the inverse tangent of both sides yields

$$\begin{aligned} \beta &= \tan^{-1}(-1.5052) \\ &\approx -56.4^\circ. \end{aligned}$$

Obviously,  $-56.4^\circ$  is not in the interval  $[0^\circ, 360^\circ)$ , but we do know that tangent is  $-$  in QII and QIV. We need to find the *positive* angles in QII and QIV that have the same reference angle as this  $\beta$ !

This is easy enough, but you may want to draw a picture to help. The positive angle in QII with reference angle  $56.4^\circ$  is

$$180^\circ - 56.4^\circ = 123.6^\circ,$$

and the positive angle in QIV with reference angle  $56.4^\circ$  is

$$360^\circ - 56.4^\circ = 303.6^\circ.$$

Therefore, the approximate solutions to this equation are  $\{123.6^\circ, 303.6^\circ\}$ .

EXAMPLE 5. Solve  $10 \cos^2 \theta - 12 \cos \theta - 7 = 0$ . Use degrees and round to the nearest hundredth.

**Solution:** This is another one of those equations that is *quadratic in form*. Unfortunately, this time the equation will not factor very easily. Thinking back to algebra, what would you do if you needed to solve a quadratic equation that did not factor “nicely”?

That’s right! You’d use the **Quadratic Formula**. Recall that for the general quadratic equation  $ax^2 + bx + c = 0$ , the solutions (roots) are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The equation  $10 \cos^2 \theta - 12 \cos \theta - 7 = 0$  is quadratic in  $\cos \theta$ , and  $a = 10$ ,  $b = -12$ , and  $c = -7$ , giving

$$\begin{aligned} \cos \theta &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(10)(-7)}}{2(10)} = \frac{12 \pm \sqrt{144 + 280}}{20} \\ &= \frac{12 \pm \sqrt{424}}{20} \\ &= \frac{12 \pm 2\sqrt{106}}{20} \\ &= \frac{6 \pm \sqrt{106}}{10}. \end{aligned}$$

This means that

$$\cos \theta = \frac{6 + \sqrt{106}}{10} \quad \text{or} \quad \cos \theta = \frac{6 - \sqrt{106}}{10}.$$

Using the arccosine function to solve the first of these equations gives

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{6 + \sqrt{106}}{10} \right) \\ &\approx \cos^{-1}(1.6296), \end{aligned}$$

which is **undefined** since 1.6296 is not in the domain of the inverse cosine function. Therefore, we get no solution from this part.

For the second equation, applying arccosine to both sides yields

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{6 - \sqrt{106}}{10} \right) \\ &\approx \cos^{-1}(-0.4296) \\ &\approx 115.44^\circ, \end{aligned}$$

which gives the QII solution. Since cosine is also  $-$  in QIII, we seek the angle in QIII that has the same reference angle as this  $\theta$ . The reference angle for  $\theta \approx 115.44^\circ$  is  $\theta' \approx 180^\circ - 115.44^\circ = 64.56^\circ$ . The angle in QIII with reference angle  $\theta' \approx 64.56^\circ$  is  $\theta \approx 180^\circ + 64.56^\circ = 244.56^\circ$ .

In this example, we were given no particular solution interval, so there are infinitely many solutions to the original equation. Using the idea of coterminal angles gives the solution set:

$$\{115.44^\circ + k \cdot 360^\circ, 244.56^\circ + k \cdot 360^\circ\},$$

where  $k$  is any integer.

EXAMPLE 6. Solve  $3 \tan 2x = -3$ .

**Solution:** In this example, we are given no particular solution interval, so we will need to find infinitely many solutions (using coterminal angles). Just for fun, let's use radians this time!

Dividing both sides by 3 gives

$$\tan 2x = -1,$$

and applying the inverse tangent to both sides yields

$$\begin{aligned} 2x &= \tan^{-1}(-1) \\ &= -\frac{\pi}{4}. \end{aligned}$$

This is the QIV solution, but we know that tangent is also negative in QII. The reference angle for  $-\pi/4$  is  $\pi/4$ , so the angle in QII with the same reference angle as this angle is  $\pi - \pi/4 = 3\pi/4$ .

So far, we have that

$$2x = -\frac{\pi}{4} + k \cdot 2\pi \quad \text{or} \quad 2x = \frac{3\pi}{4} + k \cdot 2\pi,$$

where  $k$  is any integer. Of course, this means that

$$x = -\frac{\pi}{8} + k \cdot \pi \quad \text{or} \quad x = \frac{3\pi}{8} + k \cdot \pi,$$

where  $k$  is any integer.

Therefore, the solutions to the original equation are

$$\left\{ -\frac{\pi}{8} + k \cdot \pi, \frac{3\pi}{8} + k \cdot \pi \right\},$$

where  $k$  is any integer.