Section 6.2: Trigonometric Identities (Part II)

Key points:

- Cofunction Identities: p. 547-9
- Double-Angle Identities: p. 549-51
- Given the value of a trigonometric function at some angle, be able to find the values of the trigonometric functions at twice that angle; See Example 3 in the book: p. 550
- Power-Reducing Identities: p. 551
- Half-Angle Identities: p. 551-3
- Given the value of a trigonometric function at some angle, be able to find the values of the trigonometric functions at half that angle; See Example 5 in the book: p. 552

Cofunction Identities

The cofunction identities follow directly from the sum and difference identities and the reciprocal identities.

Double-Angle Identities

The double-angle identities follow directly from the sum identities for sine, cosine, and tangent.

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{1}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned} \tag{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{3}$$

For $\sin 2\theta$, use the fact that $2\theta = \theta + \theta$ in the sine sum identity:

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta.\end{aligned}$$

For the first form of $\cos 2\theta$, use the fact that $2\theta = \theta + \theta$ in the cosine sum identity:

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta.\end{aligned}$$

The second equivalent form of $\cos 2\theta$ can be found by substituting the rewritten Pythagorean Identity $\cos^2\theta = 1 - \sin^2\theta$ into the first form:

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= (1 - \sin^2\theta) - \sin^2\theta\\ &= 1 - 2\sin^2\theta.\end{aligned}$$

The third equivalent form of $\cos 2\theta$ can be found by substituting the rewritten Pythagorean Identity $\sin^2\theta = 1 - \cos^2\theta$ into the first form:

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1 - \cos^2\theta)\\ &= 2\cos^2\theta - 1.\end{aligned}$$

For $\tan 2\theta$, use the fact that $2\theta = \theta + \theta$ in the tangent sum identity:

$$\begin{aligned}\tan 2\theta &= \tan(\theta + \theta)\\ &= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}\\ &= \frac{2\tan\theta}{1 - \tan^2\theta}.\end{aligned}$$

Power-Reducing Identities

These will not be used very often, but they are needed to get the half-angle identities.

Solve the second form of the cosine double-angle identity for $\sin^2\theta$ to get the power-reducing identity for sine:

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}.$$

Solve the third form of the cosine double-angle identity for $\cos^2\theta$ to get the power-reducing identity for cosine:

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}.$$

The power-reducing identity for tangent is obtained by dividing the power-reducing identity for sine by the power-reducing identity for cosine:

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\cos 2\theta}{1+\cos 2\theta}.$$

Half-Angle Identities

These come directly from the power-reducing identities. Choose whether to use $+$ or $-$ by finding the quadrant in which the half-angle lies.

For the sine half-angle formula, replace θ with $\frac{x}{2}$ in the power-reducing identity for sine and then take the square root of both sides:

$$\begin{aligned}
\sin^2\theta &= \frac{1-\cos 2\theta}{2} \\
\sin^2\frac{x}{2} &= \frac{1-\cos 2\left(\frac{x}{2}\right)}{2} \\
\sin^2\frac{x}{2} &= \frac{1-\cos x}{2} \\
\sqrt{\sin^2\frac{x}{2}} &= \sqrt{\frac{1-\cos x}{2}} \\
\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}}.
\end{aligned}$$

For the cosine half-angle formula, replace θ with $\frac{x}{2}$ in the power-reducing identity for cosine and then take the square root of both sides:

$$\begin{aligned}
\cos^2\theta &= \frac{1+\cos 2\theta}{2} \\
\cos^2\frac{x}{2} &= \frac{1+\cos 2\left(\frac{x}{2}\right)}{2} \\
\cos^2\frac{x}{2} &= \frac{1+\cos x}{2} \\
\sqrt{\cos^2\frac{x}{2}} &= \sqrt{\frac{1+\cos x}{2}} \\
\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}}.
\end{aligned}$$

There are 3 equivalent forms of the tangent half-angle identity. The first is obtained by dividing the half-angle identity for sine by the half-angle identity for cosine:

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

The other two forms of the tangent half-angle formula are

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$$

and

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}.$$

Example 1. Suppose $\cos\theta = \frac{5}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Find the exact value of $\tan\frac{\theta}{2}$.

Since $\frac{3\pi}{2} < \theta < 2\pi$, we know that $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$. (Just divide everything by 2!) This means that $\frac{\theta}{2}$ is in Quadrant II, and tangent is negative in Quadrant II. Using the first form of the tangent half-angle identity, we get

$$\begin{aligned}\tan\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} &= -\sqrt{\frac{1-\frac{5}{13}}{1+\frac{5}{13}}}\\ &= -\sqrt{\frac{\frac{8}{13}}{\frac{18}{13}}}\\ &= -\sqrt{\frac{8}{13}\cdot\frac{13}{18}}\\ &= -\sqrt{\frac{4}{9}}\\ &= -\frac{2}{3}\end{aligned}$$