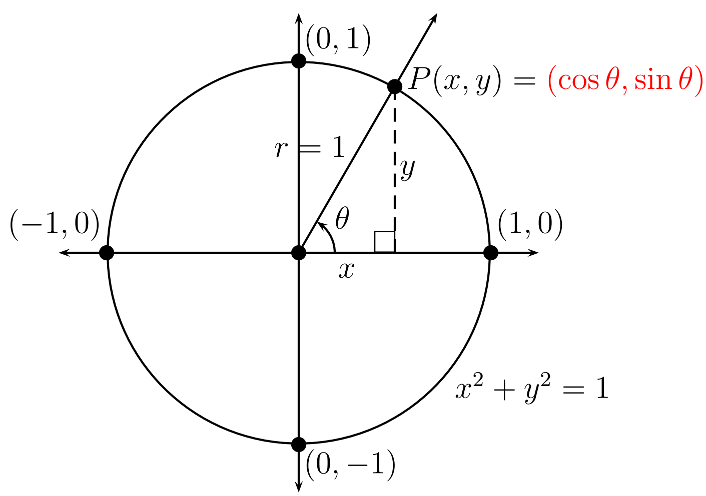
Section 5.5: Circular Functions

Key points:

- Use the unit circle to find exact values of the 6 trig functions.
- Use technology to find approximate values of the 6 trig functions.
- Given a point on the unit circle, determine the coordinates of its reflections across the $x-$axis, the $y-$axis, and the origin.
- Graph the 6 trig functions and state their properties.

The Unit Circle

Recall the unit circle:



The values of the 6 trig function for any **central angle** θ in a unit circle are given by:

$$\sin\theta = y \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}\,(y \neq 0)$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}\,(x \neq 0)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}\,(x \neq 0) \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}\,(y \neq 0)$$

A reference triangle in a unit circle has hypotenuse 1, so that the Pythagorean Theorem becomes

$$x^2 + y^2 = 1,$$

which is also the equation of the unit circle.

This is not "new"! We have already seen this in **Section 5.3** and **5.4**.

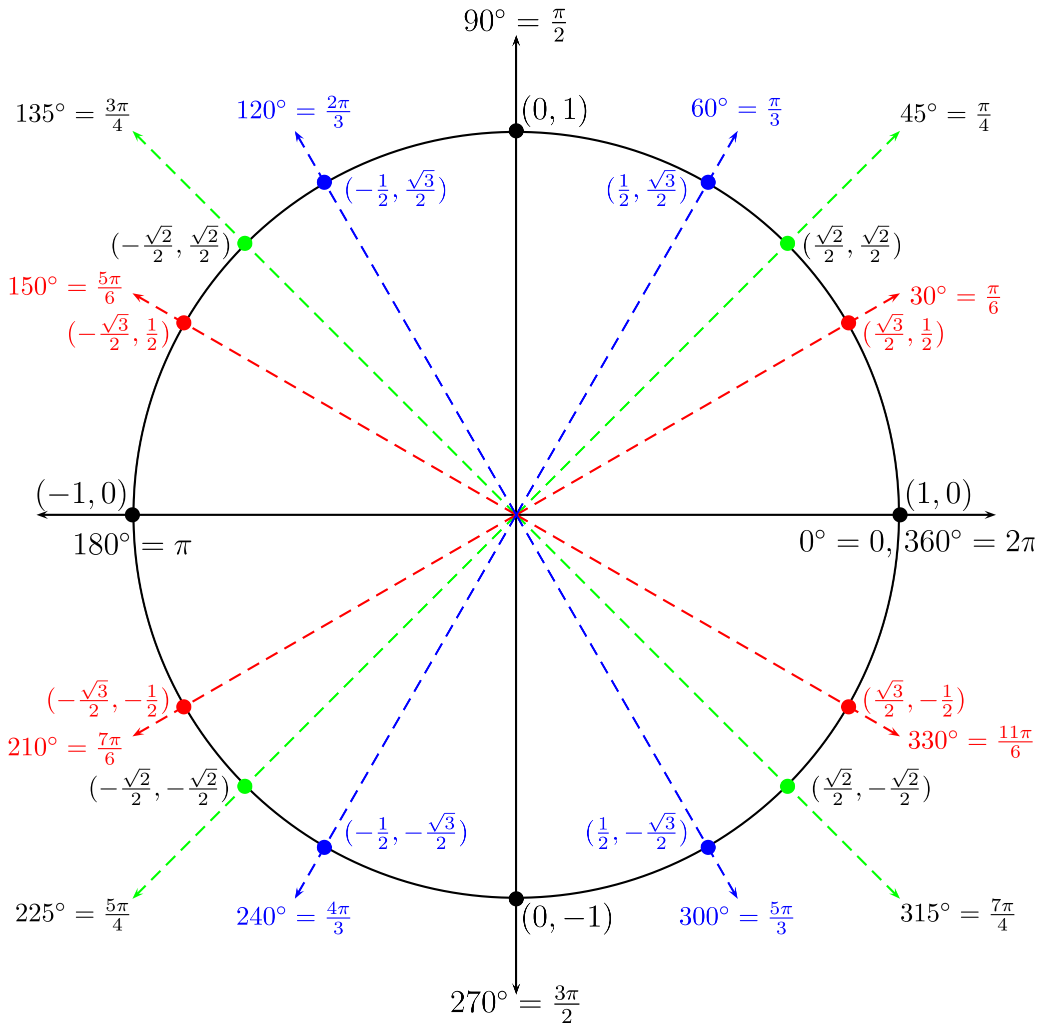


FIGURE 1. This unit circle contains the exact values of the trig functions at the quadrantal angles $0° = 0$, $90° = \pi/2$, $180° = \pi$, $270° = 3\pi/2$, and $360° = 2\pi$, and at all of the angles with reference angles $30° = \pi/6$, $45° = \pi/4$, or $60° = \pi/3$. The angles in red have reference angle $30° = \pi/6$. The angles in green have reference angle $45° = \pi/4$. The angles in blue have reference angle $60° = \pi/3$. Notice the symmetry of the angles in red, green, and blue, respectively, about the unit circle.

Recall: For any circle, the measure of a central angle θ (in radian measure) is

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}.$$

But in a unit circle $r = 1$, so

$$\theta = s.$$

That is, the measure of a central angle in a unit circle is equal to the length of the arc subtended by that central angle. Again, this is not "new"–it is simply a particular case of a concept presented in **Section 5.4**.

Example 1. Find the values of (a) $\sin(-\frac{\pi}{3})$, (b) $\sec\frac{5\pi}{6}$, and (c) $\tan\frac{2\pi}{5}$.

Solution: (a) The angle $-\pi/3$ is in QIV and sine is negative in QIV. The reference angle for $-\pi/3$ is $\pi/3$. Thus,

$$\sin(-\frac{\pi}{3}) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

We can verify this by drawing the appropriate reference triangle, as we did in **Section 5.3**, or by examining the unit circle in Figure 1 since $-\pi/3 = 5\pi/3$.

(b) The angle $5\pi/6$ is in QII and secant is negative in QII since cosine is negative there. The reference angle for $5\pi/6$ is $\pi/6$, so that

$$\sec\frac{5\pi}{6} = \frac{1}{\cos\frac{5\pi}{6}} = \frac{1}{-\cos\frac{\pi}{6}} = \frac{1}{-\frac{\sqrt{3}}{2}} = \frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}.$$

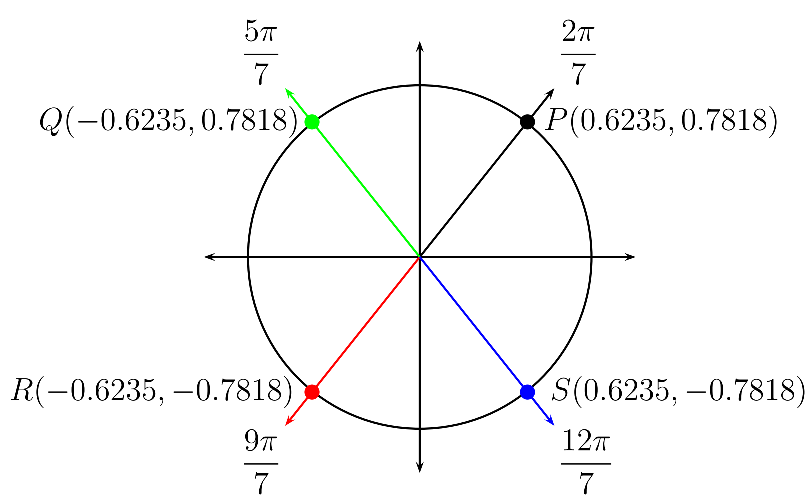
Again, we can verify this by drawing the appropriate reference triangle or by examining the unit circle in Figure 1.

(c) Since $2\pi/5 = 72°$, we do not know the exact values of the trig functions at this angle. Therefore, we need technology:

$$\tan\frac{2\pi}{5} \approx 3.07768.$$

Reflections of a Point Across the $x-$axis, the $y-$axis, and the Origin

Example 2. Locate the points of intersection of the terminal side of the angles $2\pi/7$, $5\pi/7$, $9\pi/7$, and $12\pi/7$ with the unit circle. Is there a relationship between these angles? Is there a relationship between the points of intersection?



Solution: In degrees, these angles are approximately $2\pi/7 \approx 51.4°$, $5\pi/7 \approx 128.6°$, $9\pi/7 \approx 231.4°$, and $12\pi/7 \approx 308.6°$. Moreover, $\cos(2\pi/7) \approx 0.6235$ and $\sin(2\pi/7) \approx 0.7818$.

From this figure, it is rather easy to see that the angles $2\pi/7$, $5\pi/7$, $9\pi/7$, and $12\pi/7$ all have the **same reference angle**, namely, $\theta' = 2\pi/7$.

In addition, the points of intersection of the terminal side of these angles with the unit circle appear to be symmetric about the unit circle! Relative to the point $P(0.6235, 0.7818)$, the point $Q(-0.6235, 0.7818)$ is the **reflection of P across the $y-$axis**, the point $R(-0.6235, -0.7818)$ is the **reflection of P across the origin**, and the point $S(0.6235, -0.7818)$ is the **reflection of P across the $x-$axis**.

Graphs of the Trig Functions

Study pages 496-504, especially the *Connecting the Concepts.* In addition to being able to graph the 6 basic trig functions, you will be expected to know the properties of the sine and cosine functions (page 501) and the properties of the tangent, cotangent, secant, and cosecant functions (page 504).

Important properties for each function include the domain, range, period, amplitude, whether the function is even or odd, and the location of any vertical asymptotes for the graph.

You will use these basic graphs and properties again in the next section, **Section 5.6**, which deals primarily with transformations of the sine and cosine graphs.