Section 5.4: Radians, Arc Length, and Angular Speed

Key points:

- Find the length of an arc of a circle; find the measure of a central angle of a circle.
- Convert between degree measure and radian measure.
- Find complementary and supplementary angles using radian measure.
- Find coterminal angles using radian measure.
- Find reference angles using radian measure.
- Convert between linear speed and angular speed.

Radian Measure

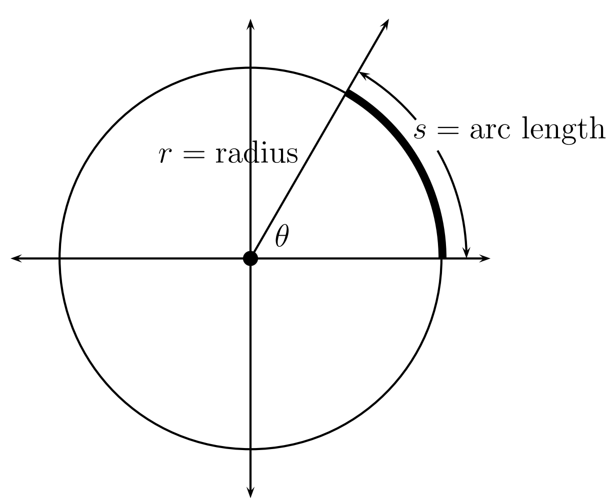
The **radian measure** of an angle of rotation θ is given by

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}.$$

Similarly, we may express this as

$$s = r\,\theta,$$

where the **central angle** θ is understood to be in radian measure. If θ is given in degrees, then it must be converted to radian measure (more about this comes later). We may also say that the arc of length s **subtends** the central angle θ.



Recall: The **circumference** of a circle with radius r is given by $C = 2\pi r$. Then the circumference of the unit circle is $C = 2\pi$.

This means that the *distance* around the unit circle is 2π. In addition, *one complete rotation* is $360°$. This gives a relationship between the distance around the unit circle and the measure of the central angle formed by one complete rotation of the unit circle:

$$2\pi = 360° \Longrightarrow \pi = 180°.$$

Conversions Between Radians and Degrees

From now on, if an angle is given and the degree symbol (°) is not present, that angle is assumed to be in radian measure!

To convert from degree measure to radian measure, multiply by $\dfrac{\pi}{180°}$.

To convert from radian measure to degree measure, multiply by $\dfrac{180°}{\pi}$.

Example 1. Convert: (a) $60°$ and (b) $75°$ to radian measure and (c) $\dfrac{7\pi}{4}$ and (d) -0.4 to degree measure.

Solution:

$$\text{(a) } 60° \times \frac{\pi}{180°} = \frac{\pi}{3}$$

$$\text{(b) } 75° \times \frac{\pi}{180°} = \frac{5\pi}{12}$$

$$\text{(c) } \frac{7\pi}{4} \times \frac{180°}{\pi} = 315°$$

$$\text{(d) } -0.4 \times \frac{180°}{\pi} = \frac{-72°}{\pi} \approx -22.92°$$

Example 2. In a circle with a 10 foot radius, an arc 140 inches long subtends a central angle θ. Find θ in radians and degrees.

Solution: All measures must be in the same units, so convert the radius to inches: 10 feet equals 120 inches. Then in radian measure,

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{140}{120} = \frac{7}{6} \approx 1.167.$$

Converting θ to degrees gives

$$\theta = \frac{7}{6} \times \frac{180°}{\pi} = \frac{210°}{\pi} \approx 66.845°.$$

Complementary Angles: Two positive angles α and β in degree measure are complementary if $\alpha + \beta = 90°$. Therefore, two positive angles α and β in radian measure are complementary if

$$\alpha + \beta = \frac{\pi}{2}.$$

Supplementary Angles: Two positive angles α and β in degree measure are supplementary if $\alpha + \beta = 180°$. Therefore, two positive angles α and β in radian measure are complementary if

$$\alpha + \beta = \pi.$$

EXAMPLE 3. Find the complement and supplement of $\theta = \frac{\pi}{6}$.

Solution: The complement α of $\theta = \frac{\pi}{6}$ satisfies

$$\frac{\pi}{6} + \alpha = \frac{\pi}{2} \Longrightarrow \alpha = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

and the supplement β of $\theta = \frac{\pi}{6}$ satisfies

$$\frac{\pi}{6} + \beta = \pi \Longrightarrow \beta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Coterminal Angles: Given an angle θ, $\theta + k \cdot 360°$ or $\theta + k \cdot 2\pi$, where k is any integer, represents all of the angles coterminal with θ.Be sure to note whether you are in degree measure or radian measure!

EXAMPLE 4. Find the smallest positive angle that is coterminal with $\theta = -\frac{9\pi}{2}$.

Solution: Just keep adding 2π until we get a positive angle:

$$-\frac{9\pi}{2} + 2\pi = -\frac{5\pi}{2} \qquad \text{(still negative)}$$

$$-\frac{5\pi}{2} + 2\pi = -\frac{\pi}{2} \qquad \text{(still negative)}$$

$$-\frac{\pi}{2} + 2\pi = \frac{3\pi}{2} \qquad \text{(stop!)}$$

Each of the angles $-5\pi/2$, $-\pi/2$, and $3\pi/2$ is coterminal with $\theta = -9\pi/2$, but $3\pi/2$ is the smallest positive angle that is coterminal with θ.

Reference Angles: Given an angle θ in radian measure, to find the reference angle θ', first convert θ to degrees, compute the reference angle, then convert that reference angle back to radians.

Example 5. Find the reference angle for $\theta = -\dfrac{3\pi}{4}$.

Solution: Converting θ to degrees gives

$$-\frac{3\pi}{4} \times \frac{180^\circ}{\pi} = -135^\circ.$$

The reference angle for -135° is 45°, which is $\pi/4$ in radian measure. Therefore, the reference angle for $\theta = -3\pi/4$ is $\theta' = \pi/4$.

Linear and Angular Speed

Linear Speed: distance traveled per unit time:

$$\begin{aligned} \text{linear speed} &= \frac{\text{distance}}{\text{time}} \\ \nu &= \frac{s}{t}. \end{aligned}$$

Angular Speed: amount of rotation per unit time:

$$\begin{aligned} \text{angular speed} &= \frac{\text{central angle}}{\text{time}} \\ \omega &= \frac{\theta}{t}. \end{aligned}$$

Recall that

$$\text{arc length} = \text{radius} \times \text{central angle}$$

$$\Longrightarrow s = r\,\theta.$$

Then

$$\nu = \frac{s}{t} = \frac{r\,\theta}{t} = r\left(\frac{\theta}{t}\right) = r\,\omega,$$

so that

$$\text{linear speed} = \text{radius} \times \text{angular speed}$$

$$\Longrightarrow \nu = r\,\omega.$$

This gives a relationship between linear speed and angular speed.

Example 6. The paddle wheel of a paddle steamer has a 30 foot diameter. John counts the revolutions and finds that the paddle wheel makes 22 revolutions per minute. If there is no current, what is the speed of the steamer, in miles per hour?

Solution: If the diameter of the paddle wheel is 30 feet, then the radius is 15 feet.

The angular speed of the paddle wheel is

$$\omega = \frac{22\ \cancel{rev}}{\text{min}} \times \frac{2\pi}{\cancel{rev}} = \frac{44\pi}{\text{min}}.$$

Thus, the linear speed of the steamer is

$$\nu = r\,\omega = 15\ \text{ft} \times \frac{44\pi}{\text{min}} = 660\pi\frac{\text{ft}}{\text{min}},$$

which we convert to miles per hour as follows:

$$\frac{660\pi\,\cancel{ft}}{\cancel{min}} \times \frac{60\ \cancel{min}}{1\ \text{hr}} \times \frac{1\ \text{mi}}{5280\,\cancel{ft}} = \frac{39600\pi\ \text{mi}}{5280\ \text{hr}} \approx 23.6\ \text{mph}.$$

Therefore, the paddle steamer is traveling about 22.6 mph.

Example 7. A rear wheel on a tractor has a 27 inch radius. Find the angle (in radians) through which a wheel rotates in 15 seconds if the tractor is traveling at 16 mph.

Solution: Since the radius is given in inches and time is given in seconds, we first convert the linear speed of 16 mph to inches per second, as follows:

$$\nu = \frac{16\ \cancel{mi}}{\cancel{hr}} \times \frac{5280\,\cancel{ft}}{\cancel{mi}} \times \frac{12\ \text{in}}{\cancel{ft}} \times \frac{1\,\cancel{hr}}{60\ \cancel{min}} \times \frac{1\ \cancel{min}}{60\ \text{sec}} = \frac{1013760\ \text{in}}{3600\ \text{sec}} = \frac{1408\ \text{in}}{5\ \text{sec}}.$$

Since

$$\nu = r\,\omega \Longrightarrow \omega = \frac{\nu}{r}$$

$$\omega = \frac{\frac{1408\ \text{in}}{5\ \text{sec}}}{27\ \text{in}} = \frac{1408\ \cancel{in}}{5\ \text{sec}} \times \frac{1}{27\ \cancel{in}} = \frac{1408}{135\ \text{sec}}.$$

Furthermore

$$\omega = \frac{\theta}{t} \Longrightarrow \theta = \omega\,t$$

$$\theta = \frac{1408}{135\ \cancel{sec}} \times 15\ \cancel{sec} = \frac{21120}{135} = \frac{1408}{9} \approx 156.4$$

Therefore, the wheel rotates about 156.4 (radians).