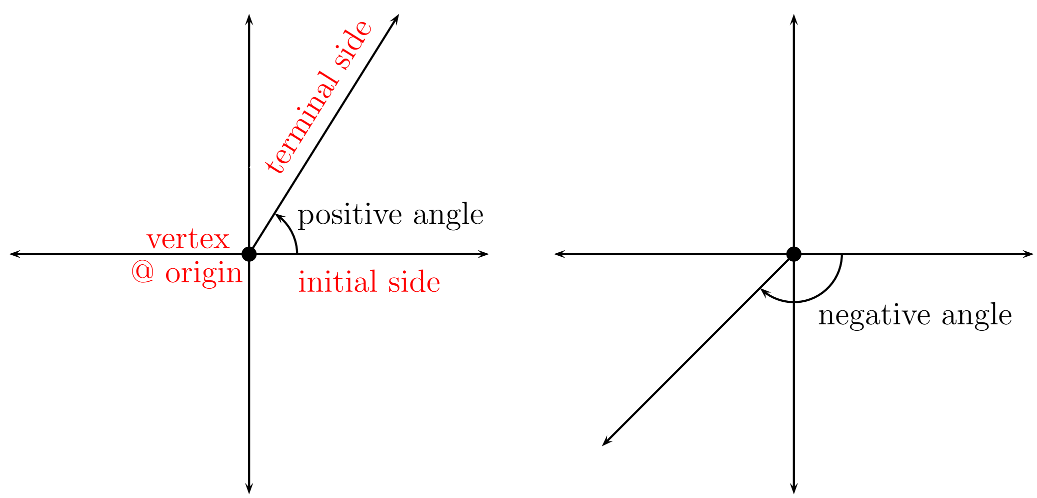
Section 5.3: Trig Functions of Any Angle

Key points:

- Draw angles in standard position.
- Find angles that are coterminal with a given angle.
- Determine the 6 trig function values for any angle in standard position when the coordinates of a point on the terminal side are given.
- Determine the 6 trig function values at quadrantal angles.
- Understand and use reference triangles and reference angles.
- Use technology to find approximate function values and angles.

Note: This very well may be the *most important section* in the text concerning trigonometry! Several of the ideas presented, such as coterminal angles and reference angles, will be used *extensively* in the remainder of this course. For this reason, it is imperative that sufficient time be given to the study of this section.

Angles in Standard Position



(a) A **positive angle** is measured counterclockwise.

(b) A **negative angle** is measured clockwise.

FIGURE 1. An angle in **standard position** has its vertex at the origin. The **initial side** will always be the positive $x-$axis. The angle is measured from the initial side to the **terminal side**.

Note: We may measure angles in either **degrees** or **radians**. Some common degree measures and their equivalent radian measures are $360° = 2\pi$, $180° = \pi$, $90° = \pi/2$, $30° = \pi/6$, $45° = \pi/4$, and $60° = \pi/3$. **Complementary angles**

are two positive angles with sum $90^\circ = \pi/2$ and **supplementary angles** are two positive angles with sum $180^\circ = \pi$. We will discuss radian measure in more detail in **Section 5.4**.

Coterminal Angles

Two or more angles that share the same terminal side are called **coterminal angles**.

Given any angle θ, the angles that are coterminal to θ can be found by *adding* or *subtracting* multiples of 360°. That is,

$$\theta + k \cdot 360^\circ \qquad \text{or} \qquad \theta + k \cdot 2\pi,$$

where k is any integer, are the angles that are coterminal with θ.

EXAMPLE 1. Find a positive angle and a negative angle that are each coterminal with 135°.

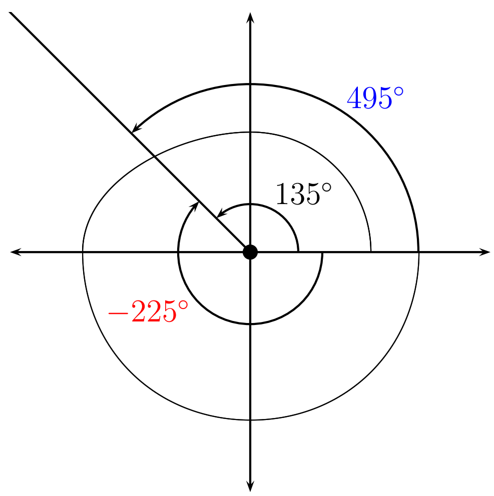


FIGURE 2. If θ is a given angle, then $\theta + k \cdot 360^\circ$ or $\theta + k \cdot 2\pi$, where k is any integer, represents all of the angles coterminal with θ. In EXAMPLE 1, $\theta = 135^\circ$ and $k = -1$ and 1. We could find other coterminal angles to 135° by setting k equal to other integer values.

Values of the Trig Functions Given a Point on the Terminal Side of an Angle

We can find the values of the 6 trig functions of any given angle if we know a point $P(x, y)$ on the terminal side of the angle, as follows:

(1) Plot the point P.
(2) Draw the terminal side from the origin through the point P.
(3) Draw a perpendicular from the point P to the "nearest" $x-$axis.
(4) This creates a **reference triangle**, as seen in FIGURE 3.

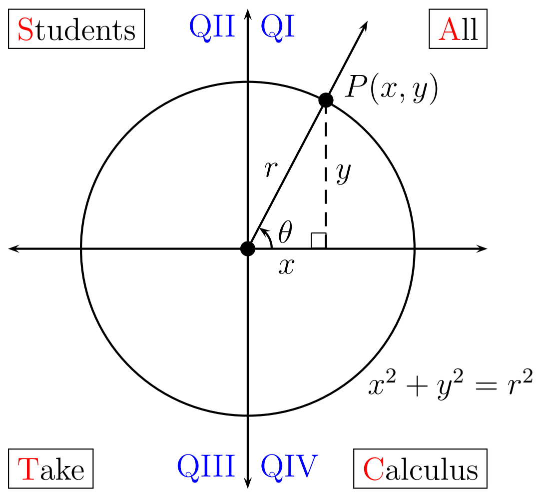


FIGURE 3. The point $P(x, y)$ is a point that is both on the terminal side of angle θ **and** on the circle of radius r centered at the origin.

FIGURE 3 contains the following information:

(1) Quadrant I (QI) contains the positive angles between $0°$ and $90°$.
(2) Quadrant II (QII) contains the positive angles between $90°$ and $180°$.
(3) QIII contains the positive angles between $180°$ and $270°$.
(4) QIV contains the positive angles between $270°$ and $360°$.
(5) "**A**ll **S**tudents **T**ake **C**alculus" is a helpful mnemonic device to help remember that
   - **A**ll trig values are positive in QI;
   - **S**in ($\Rightarrow$ csc) is positive in QII and everything else is negative;
   - **T**an ($\Rightarrow$ cot) is positive in QIII and everything else is negative;
   - **C**os ($\Rightarrow$ sec) is positive in QIV and everything else is negative.
(6) The equation of a circle centered at the origin with radius r is

$$x^2 + y^2 = r^2.$$

This also happens to give the Pythagorean Theorem for the right triangle formed by the sides x, y, and r in FIGURE 3.
(7) The values of the 6 trig functions at any angle θ can be found by applying the definitions of the trig ratios to the reference triangle as

follows:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}\ (y \neq 0)$$

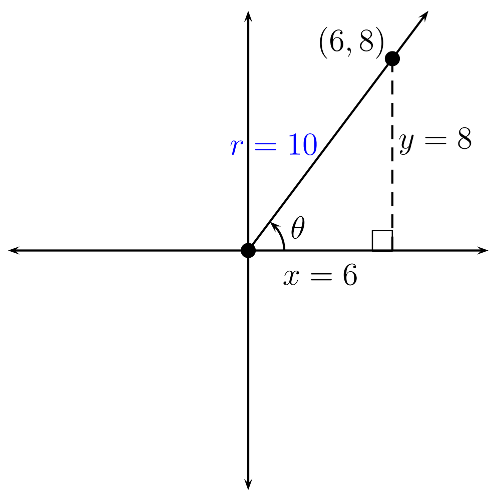
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}\ (x \neq 0)$$

$$\tan\theta = \frac{y}{x}\ (x \neq 0) \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}\ (y \neq 0)$$

(8) Reference triangles may be in any quadrant; this depends on the location of the point P.

EXAMPLE 2. Suppose the point $(6, 8)$ is on the terminal side of an angle θ. Find the values of the 6 trig functions at θ.

Solution: First draw the reference triangle and label the sides appropriately:



The hypotenuse is found using the Pythagorean Theorem:

$$r^2 = x^2 + y^2 \Longrightarrow r = \sqrt{6^2 + 8^2} = 10.$$

The rest is easy!

$$\sin\theta = \frac{y}{r} = \frac{8}{10} = \frac{4}{5} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{6}{10} = \frac{3}{5} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{8}{6} = \frac{4}{3} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{3}{4}$$

Quadrantal Angles

The terminal side of an angle may lie one of the axes, such as 180° or −90°. These angles are called **quadrantal angles**. The values of the 6 trig functions are easy to remember for the quadrantal angles if we consider the **unit circle**, that is, the circle centered at the origin with radius $r = 1$. This is just a *special case* of what we've learned already, just with $r = 1$, as seen in FIGURE 4.

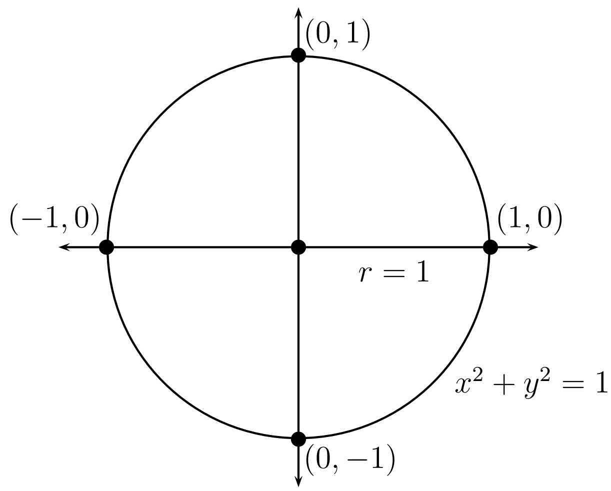


FIGURE 4. The unit circle has equation $x^2+y^2 = 1$. The points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ are the points on the unit circle that also lie on the axes.

EXAMPLE 3. Find the value of (a) $\sin 90°$, (b) $\cos 180°$, and (c) $\tan(-90°)$.

Solution: The value of

$$\sin 90° = \frac{y}{r} = 1;$$

the value of

$$\cos 180° = \frac{x}{r} = -1;$$

and the value of

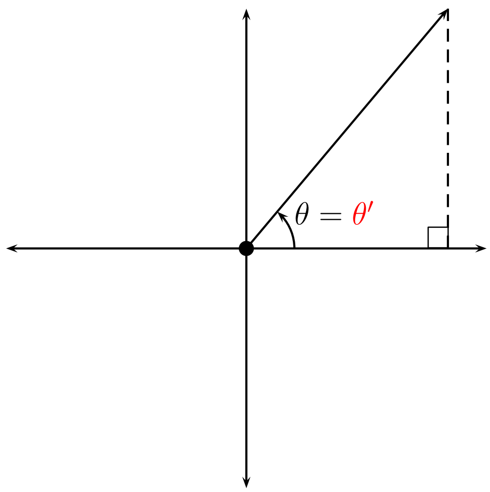
$$\tan(-90°) = \frac{y}{x} = \frac{-1}{0} = \text{undefined}.$$

Reference Angles

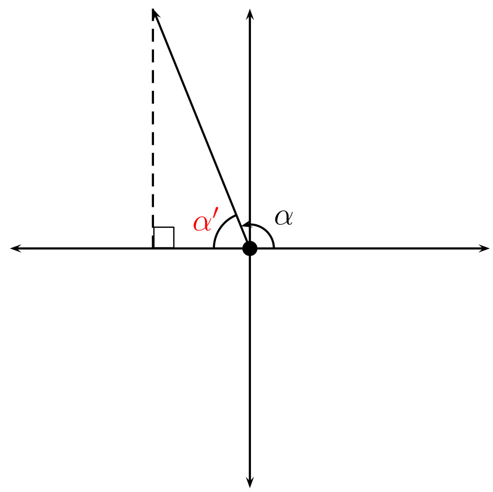
Given any angle θ, the **reference angle** for θ is the acute angle θ' ($0° < \theta' < 90°$) formed by the terminal side of θ and the "nearest" x−axis. After drawing the given angle θ, the reference angle can be easily found by drawing the appropriate reference triangle. The idea of reference angles is an extremely important concept in trig!

Example 4. Find the reference angle for (a) $\theta = 50°$, (b) $\alpha = 112°$, (c) $\beta = 225°$, and (d) $\gamma = 300°$.

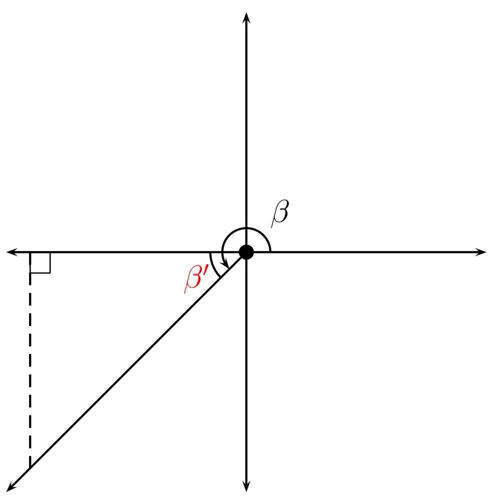
Solution:



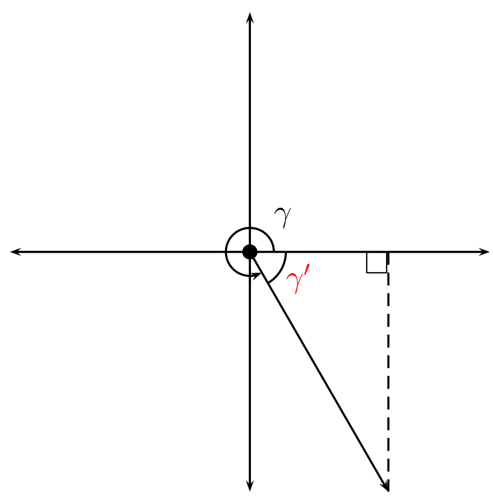
(a) The reference angle for the Quadrant I angle $\theta = 50°$ is $\theta' = 50°$.



(b) The reference angle for the Quadrant II angle $\alpha = 112°$ is $\alpha' = 180° - 112° = 68°$.



(c) The reference angle for the Quadrant III angle $\beta = 225°$ is $\beta' = 225° - 180° = 45°$.



(d) The reference angle for the Quadrant IV angle $\gamma = 300°$ is $\gamma' = 360° - 300° = 60°$.

Figure 5. It is fairly easy to find the reference angle for a given angle–simply draw the appropriate reference triangle!

Very Important Idea #1: Given any angle θ, the angles coterminal with θ are given by $\theta + k \cdot 360°$ or $\theta + k \cdot 2\pi$, where k is any integer. Every angle that is coterminal with θ has the **same reference angle**!

Very Important Idea #2: Given any angle θ, the reference angle θ' is the acute angle formed by the terminal side of θ and the "nearest" $x-$axis. The **value of a trig function at any angle** θ is EITHER

(a) equal to the *value at the reference angle*

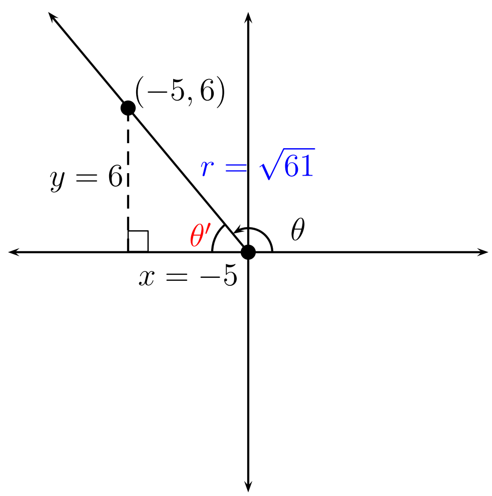
OR

(b) equal to the *negative of the value at the reference angle.*

We decide which of either (a) or (b) above is true based on the quadrant in which the given angle θ lies.

EXAMPLE 5. The point $(-5, 6)$ is on the terminal side of an angle θ. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$.

Solution: Draw the reference triangle, locate the reference angle, and label the sides appropriately:



The hypotenuse is found using the Pythagorean Theorem:

$$r^2 = x^2 + y^2 \Longrightarrow r = \sqrt{(-5)^2 + 6^2} = \sqrt{61}.$$

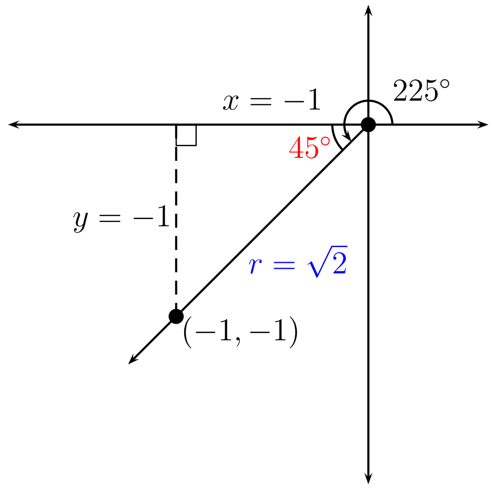
The rest is easy!

$$\sin\theta = \frac{6}{\sqrt{61}} = \frac{6\sqrt{61}}{61} \qquad \cos\theta = \frac{-5}{\sqrt{61}} = \frac{-5\sqrt{61}}{61} \qquad \tan\theta = \frac{6}{-5} = \frac{-6}{5}.$$

Notice that we found $\sin\theta$, $\cos\theta$, and $\tan\theta$ using the reference triangle and the reference angle, and chose $+$ or $-$ based on the fact that θ was in Quadrant II! In Quadrant II, sine is positive, but cosine and tangent are negative.

EXAMPLE 6. Compute $\sin 225°$ and $\cos 225°$ exactly.

Solution: Draw the reference triangle and find the reference angle:

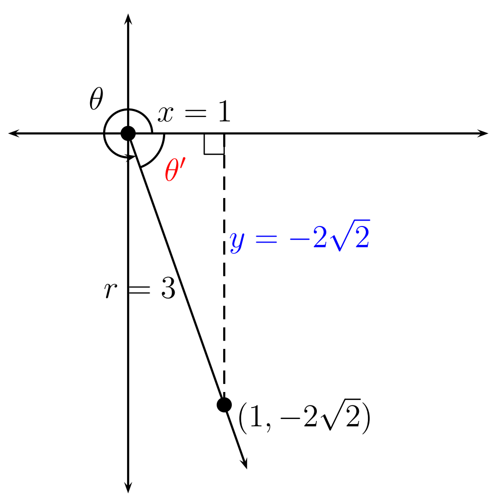


Since the reference angle for $\theta = 225°$ is $\theta' = 45°$, we can find the exact values of the trig functions at $225°$. Using a $45° - 45° - 90°$ right triangle in QIII with sides labeled appropriately, we get

$$\sin 225° = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2} \qquad \cos 225° = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}.$$

EXAMPLE 7. Suppose $\cos\theta = \frac{1}{3}$ and θ is in QIV. Find $\sin\theta$ and $\tan\theta$.

Solution: Draw the reference triangle, locate the reference angle, and label the sides appropriately:



The missing side is found using the Pythagorean Theorem:

$$x^2 + y^2 = r^2 \Longrightarrow y = -\sqrt{3^2 - 1^2} = \sqrt{8} = -2\sqrt{2}$$

since y is negative in QIV.

This gives

$$\sin\theta = \frac{-2\sqrt{2}}{3} \qquad\qquad \tan\theta = \frac{-2\sqrt{2}}{1} = -2\sqrt{2}.$$

EXAMPLE 8. Approximate $\cos\theta$ to 5 decimal places if $\theta = 142°15'48''$. What is the reference angle for θ?

Solution: Using technology, we get

$$\cos 142°15'48'' \approx -0.79083$$

and since θ is in QII, the reference angle is

$$\theta' = 180° - 142°15'48'' = 37°44'12''.$$

EXAMPLE 9. Suppose $\tan\theta = 0.246$ and θ is in the interval $(180°, 270°)$. Find θ, rounded to the nearest hundredth.

Solution: We have

$$\begin{aligned} \tan\theta = 0.246 \Longrightarrow \theta &= \tan^{-1}(0.246) \\ &\approx 13.82034°. \end{aligned}$$

Obviously, this angle is *not* in QIII. This is really the reference angle for θ! The angle θ in QIII with reference angle $13.82034°$ is

$$\theta \approx 180° + 13.82034° \approx 193.82°.$$