Section 4.5: Solving Exponential and Logarithmic Equations

Key points:

- Solve exponential equations.
- Solve logarithmic equations.

You really, *really* need to read this section in the book. Look at all of the examples and make sure that you understand them!

Solving Exponential Equations

Basically, take the logarithm of both sides. Use `log` or `ln`.

Example 1. Solve $3.6^{2x} = 19$. (Using natural log.)

$$\begin{aligned}
\ln 3.6^{2x} &= \ln 19 && \text{(take the ln of both sides)}\\
2x \ln 3.6 &= \ln 19 && \text{(bring down the exponent)}\\
x(2\ln 3.6) &= \ln 19 && \text{(factor out } x\text{)}\\
x &= \frac{\ln 19}{2\ln 3.6} && \text{(divide both sides by coef of } x\text{; exact answer)}\\
x &\approx 1.149 && \text{(approximation)}
\end{aligned}$$

Now, check the solution:

$$\begin{aligned}
3.6^{2(1.149)} &\overset{?}{\approx} 19\\
18.984 &\overset{?}{\approx} 19 && \text{(Close enough!)}
\end{aligned}$$

Example 2. Solve $28^x = 10^{-4x}$. (Using common log.)

$$\begin{aligned}
\log 28^x &= \log 10^{-4x} && \text{(take the log of both sides)}\\
x\log 28 &= -4x\log 10 && \text{(bring down the exponent)}\\
x\log 28 &= -4x && \text{(since } \log 10 = 1\text{)}\\
x\log 28 + 4x &= 0 && \text{(move } x\text{'s to one side, numbers to the other)}\\
x(\log 28 + 4) &= 0 && \text{(factor out } x\text{; this is } \textbf{NOT } \log 32!\text{)}\\
x &= \frac{0}{\log 28 + 4} = 0 && \text{(divide both sides by coef of } x\text{)}
\end{aligned}$$

Now, check the solution:

$$28^0 \stackrel{?}{=} 10^{-4(0)} \qquad \text{(Yes, both sides equal 1.)}$$

Example 3. Solve $6^{3x+1} = 5^{2x+3}$.

$$\begin{aligned}
\ln 6^{3x+1} &= \ln 5^{2x+3} && \text{(take the ln of both sides)} \\
(3x+1)\ln 6 &= (2x+3)\ln 5 && \text{(bring down the exponent; use "()"!)} \\
3x\ln 6 + \ln 6 &= 2x\ln 5 + 3\ln 5 && \text{(distribute)} \\
3x\ln 6 - 2x\ln 5 &= 3\ln 5 - \ln 6 && \text{(move } x\text{'s to one side, numbers to other)} \\
x(3\ln 6 - 2\ln 5) &= 3\ln 5 - \ln 6 && \text{(factor out } x\text{)} \\
x &= \frac{3\ln 5 - \ln 6}{3\ln 6 - 2\ln 5} && \text{(divide by coef of } x\text{; exact answer)} \\
x &\approx 1.408 && \text{(approximation)}
\end{aligned}$$

Check the solution:

$$6^{3(1.408)+1} \stackrel{?}{\approx} 5^{2(1.408)+3}$$
$$11,616 \stackrel{\checkmark}{\approx} 11,620$$

Example 4. Solve $3e^{2x-7} - 5 = 3$. (Write down what was done in each step!)

$$\begin{aligned}
3e^{2x-7} &= 8 \\
e^{2x-7} &= \frac{8}{3} \\
\ln e^{2x-7} &= \ln\frac{8}{3} \\
(2x-7)\ln e &= \ln\frac{8}{3} \\
2x-7 &= \ln\frac{8}{3} \\
2x &= \ln\frac{8}{3} + 7 \\
x &= \frac{\ln\frac{8}{3}+7}{2} \approx 3.99
\end{aligned}$$

Check the solution:

$$3e^{2(3.99)-7} - 5 \overset{?}{\approx} 3$$
$$2.993 \overset{\checkmark}{\approx} 3$$

Example 5. Solve $e^x + e^{-x} = 6$.

$$\begin{aligned}
e^x + \frac{1}{e^x} &= 6 && \text{(Rewrite } e^{-x} \text{ as } \frac{1}{e^x}\text{)} \\
e^x\left(e^x + \frac{1}{e^x}\right) &= 6e^x && \text{(Multiply both sides by } e^x\text{)} \\
e^{2x} + 1 &= 6e^x && \text{(Simplify)} \\
e^{2x} - 6e^x + 1 &= 0 && \text{(Move everything to one side)}
\end{aligned}$$

This last expression is *quadratic in form.* If we let $t = e^x$, then the last equation becomes

$$t^2 - 6t + 1 = 0.$$

Using the quadratic formula to solve for t, we get

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)} = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2}$$
$$= 3 \pm 2\sqrt{2}$$

Since we wish to solve for x, we replace t with e^x and take the ln of both sides:

$$\begin{aligned}
e^x &= 3 + 2\sqrt{2} && \text{or} && e^x = 3 - 2\sqrt{2} && \\
x &= \ln(3 + 2\sqrt{2}) && \text{or} && x = \ln(3 - 2\sqrt{2}) && \text{(exact solutions)} \\
x &\approx 1.763 && \text{or} && x \approx -1.763 && \text{(approximations)}
\end{aligned}$$

Check that both solutions satisfy the original equation!

Solving Logarithmic Equations

Use either the **Definition of a Logarithm**

$$\log_a x = y \quad \text{if and only if} \quad x = a^y$$

or the **Property of Logarithmic Equality**

$$\log_a M = \log_a N \quad \text{if and only if} \quad M = N.$$

To use the definition, we must condense everything to a single logarithm on the left-hand side. To use the property of logarithmic equality, we must condense each side of the equation to a single logarithm.

Always check the solution!

Example 6. Solve $\log_8 x = 2.3$. By the definition of a logarithm, we have

$$\begin{aligned} x &= 8^{2.3} && \text{(exact answer)} \\ x &\approx 119.428 && \text{(approximation)} \end{aligned}$$

Use the change-of-base formula to check: $\log_8 119.428 = \dfrac{\ln 119.428}{\ln 8} \approx 2.3.$

Example 7. Solve $\ln(x+3) = 7$. Since ln is the log in base e, by the definition of a logarithm we have

$$\begin{aligned} x + 3 &= e^7 && \\ x &= e^7 - 3 && \text{(exact answer)} \\ x &\approx 1093.63 && \text{(approximation)} \end{aligned}$$

Check the solution:

$$\begin{aligned} \ln(1093.63 + 3) &\overset{?}{\approx} 7 \\ 6.999997 &\overset{\checkmark}{\approx} 7 \end{aligned}$$

Example 8. Solve $\log(3x + 4) = \log(x^2)$. By the property of logarithmic equality, we have

$$\begin{aligned} 3x + 4 &= x^2 && \\ 0 &= x^2 - 3x - 4 && \text{(Move everything to one side)} \\ 0 &= (x - 4)(x + 1) && \text{(Factor)} \end{aligned}$$

Thus, $x = 4$ or $x = -1$. Substituting each of these values back into the original equation gives a true statement, so both solutions are valid. **Verify this!**

Example 9. Solve $\log_5(x+4)+\log_5(x-4)=2$.

$$\log_5(x+4)(x-4)=2 \qquad \text{(Product Rule)}$$

$$\log_5(x^2-16)=2 \qquad \text{(FOIL)}$$

$$x^2-16=5^2 \qquad \text{(Definition of Logarithm)}$$

$$x^2=41 \qquad \text{(Simplify)}$$

$$x=\pm\sqrt{41}$$

$$x\approx\pm 6.403$$

Checking the solutions reveals that $x=\sqrt{41}\approx 6.403$ satisfies the original equation but $x=-\sqrt{41}\approx -6.403$ does not.