

Section 4.4: Properties of Logarithms

Key points:

- Review the notes for **Section 4.3: Logarithmic Functions** along with these notes.
- Read logarithmic expressions.
- Understanding logarithmic expressions.
- Expand logarithmic expressions using various properties of logarithms.
- Condense logarithmic expressions using various properties of logarithms.

How to Read Logarithmic Expressions

When you see $\log_a x$, you read this as “the log(arithm) in base a of x ”. In **Section 4.3**, we learned that for logarithms, we require $x > 0$ and $a > 0$, but $a \neq 1$.

Example 1. $\log_3 9$ is read “the log in base 3 of 9”.

Example 2. $\log_5 125$ is read “the log in base 5 of 125”.

Example 3. $\log_{17} 25.3$ is read “the log in base 17 of 25.3”.

Example 4. $\log_{9.32} 47.8$ is read “the log in base 9.32 of 47.8”.

Example 5. $\log 32$ is read “the **common logarithm** of 32”. A logarithm with no base given is understood to be base 10. Usually, we would not write $\log_{10} 32$.

Example 6. $\ln 17.25$ is read “the **natural logarithm** of 17.25”. A natural logarithm is understood to be base e , which is approximately 2.71828. Usually, we would not write $\log_e 17.25$.

Understanding the Meaning of Logs

When you see $\log_a x$, ask yourself “ a to what power is x ?” What this is really saying is that when you are computing/evaluating a logarithmic expression, you are trying to find an exponent. In other words, **a logarithm is an exponent!**

Example 7. $\log_3 9$ is asking “3 to what power is 9?” Of course $3^2 = 9$, so $\log_3 9 = 2$.

Example 8. $\log_5 125$ is asking “5 to what power is 125?” Well $5^3 = 125$, so $\log_5 125 = 3$.

Before doing the next two examples, recall the **Change-of-Base Formula**:

$$\log_a M = \frac{\log_b M}{\log_b a} = \frac{\log M}{\log a} = \frac{\ln M}{\ln a}.$$

What this says that we may change a logarithm in base a to **ANY** base b that we like, so we choose **either** base 10 **or** base e .

Example 9. $\log_{17} 25.3$ is asking “17 to what power is 25.3?” Using the change-of-base formula with common log, this becomes

$$\log_{17} 25.3 = \frac{\log 25.3}{\log 17},$$

which we type in the Accessories Calculator as the following sequence of keystrokes: 25.3, [**log**], [/], 17, [**log**], [=]. We should get that

$$\log_{17} 25.3 \approx 1.14033.$$

This means that

$$17^{1.14033} \approx 25.3.$$

Try it for yourself! Note that we could have used [**ln**] instead of [**log**].

Example 10. $\log_{9.32} 47.8$ is asking “9.32 to what power is 47.8?” Using the change-of-base formula with natural log this time, we get

$$\log_{9.32} 47.8 = \frac{\ln 47.8}{\ln 9.32},$$

which we enter in the Accessories Calculator as follows: 47.8, [**ln**], [/], 9.32, [**ln**], [=]. We should get that

$$\log_{9.32} 47.8 \approx 1.73241.$$

This means that

$$9.32^{1.73241} \approx 47.8.$$

Try it for yourself! Note that we could have used $[\log]$ instead of $[\ln]$.

Example 11. $\log 32$ is asking “10 to what power is 32?” Since $10^1 = 10$ and $10^2 = 100$, we know that $\log 32$ is between 1 and 2. To approximate this value using the Accessories Calculator, type 32, then $[\log]$, to get that

$$\log 32 \approx 1.50515.$$

This means that $10^{1.50515} \approx 32$.

Example 12. $\ln 17.25$ is asking “ e to what power is 17.25?” Well, $e^1 = e \approx 2.7$, $e^2 \approx 7.4$, and $e^3 \approx 20.1$, so we know that $\ln 17.25$ is between 2 and 3. To approximate this value using the Accessories Calculator, type 17.25, followed by $[\ln]$, to get that

$$\ln 17.25 \approx 2.84781.$$

This means that $e^{2.84781} \approx 17.25$.

Note: We actually can use the change-of-base formula and Accessories Calculator for Example 7 and Example 8, also. That is,

$$\log_3 9 = \frac{\log 9}{\log 3} = \frac{\ln 9}{\ln 3} = 2$$

and

$$\log_5 125 = \frac{\log 125}{\log 5} = \frac{\ln 125}{\ln 5} = 3.$$

You may use **either** common logs **or** natural logs when evaluating change-of-base problems—you will get the same answer regardless. Try it and see!

Properties of Logs: Part I

Some logarithm problems can be solved without much fuss at all, if we remember *a logarithm is an exponent*.

(1) The log in base a of 1 is 0, since $a^0 = 1$ for all logarithmic bases a . This is a rule of exponents, so it should apply to logarithms! In other words, for all bases a ,

$$\log_a 1 = \log 1 = \ln 1 = 0.$$

Example 13. $\log_{17.4} 1 = 0$.

(2) The log in base a of a is 1, since $a^1 = a$ for all logarithmic bases a . This is a rule of exponents, so it should apply to logarithms! In other words, for all bases a ,

$$\log_a a = \log 10 = \ln e = 1.$$

Example 14. $\log_{3\sqrt{2}} 3\sqrt{2} = 1$.

(3) The log in base a of a^p is p for all logarithmic bases a . In other words, for all bases a ,

$$\log_a a^p = \log 10^p = \ln e^p = p.$$

Recall what a log means and the answer is easy! When we write $\log_a a^p$, we are asking: “ a to what power is a^p ?” Of course the answer will be p !

Example 15. $\log_{11} 11^{(x^2-3x+4)} = x^2 - 3x + 4$.

Properties of Logs: Part II

(1) **The Power Rule.** The power rule for exponents says

$$(a^m)^p = a^{mp},$$

that is, to find the power of a power, just multiply the exponents. Since a logarithm is an exponent, this becomes

$$\log_a M^p = p \log_a M,$$

where $M > 0$, a is any logarithmic base, and p is any real number (or expression). That is, the logarithm of a power of M is the exponent of M times the logarithm of M . Most people just say “you can bring the exponent to the front”.

Example 16. $\log_5 12^{3x} = 3x \log_5 12$.

Example 17. $\ln 9^{x+2} = (x+2) \ln 9 = x \ln 9 + 2 \ln 9$.

(2) **The Product Rule.** The product rule for exponents says

$$a^m \cdot a^n = a^{m+n},$$

that is, to multiply exponential expressions with the same base, simply add the exponents. Since a logarithm is an exponent, this becomes

$$\log_a MN = \log_a M + \log_a N,$$

where M and N must **both** be positive and a is any logarithmic base. That is, the logarithm of a product is the sum of the logarithms of the factors.

Example 18. Expand: $\log_t x^3 y^2$, where $x, y > 0$.

$$\begin{aligned} \log_t x^3 y^2 &= \log_t x^3 + \log_t y^2 && \text{(Product Rule)} \\ &= 3 \log_t x + 2 \log_t y && \text{(Power Rule)} \end{aligned}$$

Example 19. Condense: $\log_3(x+1) + \log_3(x-2)$, where $x > 2$.

$$\log_3(x+1) + \log_3(x-2) = \log_3(x+1)(x-2) = \log_3(x^2 - x - 2)$$

by the Product Rule.

(3) **The Quotient Rule.** The quotient rule for exponents says

$$\frac{a^m}{a^n} = a^{m-n},$$

that is, to divide exponential expressions with the same base, simply subtract the exponents. Since a logarithm is an exponent, this becomes

$$\log_a \frac{M}{N} = \log_a M - \log_a N,$$

where M and N must **both** be positive and a is any logarithmic base. That is, the logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.

Example 20. Expand: $\log_t \frac{a^5}{b^7}$, where $a, b > 0$.

$$\begin{aligned} \log_t \frac{a^5}{b^7} &= \log_t a^5 - \log_t b^7 && \text{(Quotient Rule)} \\ &= 5 \log_t a - 7 \log_t b && \text{(Power Rule)} \end{aligned}$$

Example 21. Condense: $2 \log_4 x - \frac{1}{2} \log_4 y$, where $x, y > 0$.

$$\begin{aligned} 2 \log_4 x - \frac{1}{2} \log_4 y &= \log_4 x^2 - \log_4 \sqrt{y} && \text{(Power Rule)} \\ &= \log_4 \frac{x^2}{\sqrt{y}} && \text{(Quotient Rule)} \end{aligned}$$

Example 22. Let's try something harder: Expand

$$\log_a \sqrt[3]{\frac{x^4 y^7}{a^2}},$$

where $x, y, a > 0$.

$$\begin{aligned} \log_a \sqrt[3]{\frac{x^4 y^7}{a^2}} &= \log_a \left(\frac{x^4 y^7}{a^2} \right)^{\frac{1}{3}} && \text{(Property of Exponents: } \sqrt[n]{x} = x^{\frac{1}{n}} \text{)} \\ &= \log_a \frac{x^{4/3} y^{7/3}}{a^{2/3}} && \text{(Property of Exponents: } [xy]^n = x^n y^n \text{)} \\ &= \log_a x^{4/3} y^{7/3} - \log_a a^{2/3} && \text{(Quotient Rule)} \\ &= \log_a x^{4/3} + \log_a y^{7/3} - \log_a a^{2/3} && \text{(Product Rule)} \\ &= \frac{4}{3} \log_a x + \frac{7}{3} \log_a y - \frac{2}{3} \log_a a && \text{(Power Rule)} \\ &= \frac{4}{3} \log_a x + \frac{7}{3} \log_a y - \frac{2}{3} && \text{(Property of logs: } \log_a a = 1 \text{)} \end{aligned}$$

(4) **A Base to a Logarithmic Power.** For any logarithmic base a and positive number x ,

$$a^{\log_a x} = x.$$

Example 23. $10^{\log(x^2+7)} = x^2 + 7$.

(5) **The Base-Exponent Property.** For any logarithmic base a ,

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

Example 24. Suppose $e^{4x-3} = e^{2x+1}$. Then

$$\begin{aligned}4x - 3 &= 2x + 1 \\4x - 2x &= 1 + 3 \\2x &= 4 \\x &= 2.\end{aligned}$$

To check this, we substitute $x = 2$ into the original equation:

$$e^{4(2)-3} \stackrel{?}{=} e^{2(2)+1} \quad \text{if and only if} \quad e^5 \stackrel{\checkmark}{=} e^5.$$

(6) **The Property of Logarithmic Equality.** For any $M, N > 0$ and logarithmic base a ,

$$\log_a M = \log_a N \quad \text{if and only if} \quad M = N.$$

Example 25. Suppose $\log(x^2 - 6) = \log x$. Then

$$\begin{aligned}x^2 - 6 &= x \\x^2 - x - 6 &= 0 \\(x - 3)(x + 2) &= 0\end{aligned}$$

Then $x = 3$ or $x = -2$. Substituting these values into the original equation gives

$$\begin{array}{ll}\log(3^2 - 6) \stackrel{?}{=} \log 3 & \log((-2)^2 - 6) \stackrel{?}{=} \log -2 \\ \log 3 \stackrel{\checkmark}{=} \log 3 & \log -2 \stackrel{\times}{=} \log -2\end{array}$$

Even though the second verification seems to be OK, we know that we cannot take the log of a negative number. Note, it is not necessarily because the value of x itself was negative that the second value does not work. It is because when we substituted that value into the **original equation**, we got something that did not make sense.