# Section 4.3: Logarithmic Functions

**Key points:**

- Review the notes for **Section 4.4: Properties of Logarithms** along with these notes.
- Logarithmic functions are the inverses of exponential functions.
- Convert between exponential equations and logarithmic equations.
- Graph logarithmic functions.
- Solve applied problems using logarithmic functions.

## Logarithmic Functions

A **logarithmic function** in base $a$ is given by

$$f(x) = \log_a x,$$

where $x > 0$ is any real number and $a > 0$, but $a \neq 1$.

One way to think about logarithmic equations is to interpret them as exponential equations: When you see

$$y = \log_a x,$$

think to yourself: I need to find the number $y$ such that

$$a^y = x.$$

## Properties of Logarithmic Functions

Suppose that $f(x) = \log_a x$, where $x > 0$, $a > 0$, but $a \neq 1$, is a logarithmic function. Then the following are true:

- Domain: $(0, \infty)$.
- Range: $(-\infty, \infty)$.
- A logarithmic function is **increasing** if $a > 1$.

- A logarithmic function is **decreasing** if $0 < a < 1$.
- The $x$-intercept is the point $(1, 0)$.
- The vertical asymptote is the $y$-axis, that is, the line $x = 0$.
- All logarithmic functions are $1-1$. Their inverses are called **exponential functions**, as discussed in **Section 4.2**.

## Conversions

We can convert between logarithmic equations and exponential equations using the fact that

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

Note that **a logarithm is an exponent**! We need to find this exponent!

**Example 1.** The exponential equation $81^{1/4} = 3$ can be converted to an equivalent logarithmic equation:

$$81^{1/4} = 3 \quad \text{is the same as} \quad \log_{81} 3 = \frac{1}{4}.$$

The logarithmic equation $\log_p k = t$ can be converted to an equivalent exponential equation:

$$\log_p k = t \quad \text{is the same as} \quad p^t = k.$$

## Basic Properties of Logarithms

$$\log_a 1 = 0 \quad \text{since} \quad a^0 = 1 \quad \text{for all bases } a$$

$$\log_a a = 1 \quad \text{since} \quad a^1 = a \quad \text{for all bases } a$$

## Common Logarithms

A common logarithm has base 10. The **common logarithmic function** is written as

$$f(x) = \log x.$$

If there is no base given, assume it is a common logarithm; we do not usually write $\log_{10} x$.

Some basic properties of common logs are

$$\log 1 = 0 \quad \text{since} \quad 10^0 = 1$$

$$\log 10 = 1 \quad \text{since} \quad 10^1 = 10$$

**Example 2.** Compute the value of $\log 1000$. This is a common logarithm, which has base 10, so we must ask ourselves "10 to *what power* is 1000?" Of course, $10^3 = 1000$, so we have $\log 1000 = 3$. You can compute this value using the "Accessories Calculator" on your computer by entering 1000 and then pressing the `[log]` button.

## Natural Logarithms

A natural logarithm has base $e$. The **natural logarithmic function** is written as

$$f(x) = \ln x.$$

We do not usually write $\log_e x$.

Some basic properties of natural logs are

$$\ln 1 = 0 \quad \text{since} \quad e^0 = 1$$

$$\ln e = 1 \quad \text{since} \quad e^1 = e$$

**Example 3.** Compute: $\ln 34$. Using the Accessories Calculator, type 34 and then press the `[ln]` button. You should get that

$$\ln 34 \approx 3.5264.$$

This means that $e^{3.5264} \approx 34$. Check it!

## Using a Calculator to Approximate Logarithms

We have already seen examples of how to compute the common logarithm (base 10) and the natural logarithm (base $e$). What if we need to compute something like $\log_{13} 125$? We need the Change-of-Base Formula for logarithms.

**Change-of-Base Formula:** If we need to compute the logarithm of a number in some base other than base 10 or base $e$, then we use the fact that

$$\log_a M = \frac{\log_b M}{\log_b a} = \frac{\log M}{\log a} = \frac{\ln M}{\ln a}.$$

What this says is that we may **convert** a logarithm in some base $a$ to any base $b$ that we like. For convenience, we choose the new base to be **either** base 10 **or** base $e$.

**Example 4.** Compute the value of $\log_{13} 125$. We know that $13^1 = 13$ and $13^2 = 169$, which means that $\log_{13} 125$ is between 1 and 2. Using the change-of-base formula, we rewrite

$$\log_{13} 125 \quad \text{as} \quad \frac{\log 125}{\log 13},$$

and we can now use the Accessories Calculator to approximate this. Enter 125, press the `[log]` button, press the divide button `[/]`, enter 13, press the `[log]` button, and then press the equals button `[=]`. You should get that

$$\log_{13} 125 \approx 1.8824.$$

We could have also worked this problem by replacing `[log]` by `[ln]`. Make sure to use one or the other, and do not mix `[log]` and `[ln]` in the same problem!