Section 4.1: Inverse Functions

Key points:

- Find the inverse of a relation.
- Determine whether a function is $1-1$.
- Use the Horizontal Line Test.
- Given a $1-1$ function, find its inverse.
- Simplify expressions of the type $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.

Inverse of a Relation

To find the inverse of a relation, **switch the x's and y's**. If a graph is given, simply **reflect the graph over the line $y = x$**.

Example 1. The inverse of the relation

$$\{(-2,4),(3,-7),(8,2)\} \quad \text{is} \quad \{(4,-2),(-7,3),(2,8)\}.$$

The inverse of the relation $y = x^4 - x^2 + 3$ is $x = y^4 - y^2 + 3$.

One-to-One Functions

- Abbreviated as "$1-1$".
- A function f is $1-1$ if different inputs yield different outputs:

$$\text{if } a \neq b, \text{ then } f(a) \neq f(b).$$

- A function f is $1-1$ if when the outputs are the same, the inputs are the same:

$$\text{if } f(a) = f(b), \text{ then } a = b.$$

- If a relation is **NOT** a function, then the relation **CANNOT** be $1-1$!
- Every $1-1$ function f has an inverse, denoted f^{-1}. The little -1 is **NOT** an exponent! The symbol $f^{-1}(x)$ is read as "f inverse of x" or as "the inverse of $f(x)$."

Properties of $1-1$ Functions

- If a function f is increasing over its entire domain, then f is $1-1$.
- If a function f is decreasing over its entire domain, then f is $1-1$.
- If a function f is $1-1$, then its inverse f^{-1} is a function.
- The domain of a $1-1$ function f is the range of f^{-1}.
- The range of a $1-1$ function f is the domain of f^{-1}.
- The graph of f^{-1} is the reflection of the graph of f over the line $y=x$.

The Horizontal Line Test

Suppose f is a function whose graph is known. If **ANY horizontal line** can be drawn to intersect the graph of f **more than once**, then the function is **NOT** $1-1$.

Finding Equations for Inverses

If f is a $1-1$ function (use Horizontal Line Test and Vertical Line Test to check this), then its inverse can be found by completing the following steps:

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.

Composition of Inverse Functions

If f is a $1-1$ function, then f^{-1} is the *unique* function such that **both** of the following properties hold:

1. $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
2. $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for all x in the domain of f.