Section 3.6: Polynomial and Rational Inequalities

Key points:

- Interpret expressions involving inequalities.
- Solve polynomial inequalities.
- Solve rational inequalities.

Interpreting Inequalities

We have spent a lot of time talking about *solving equations*. In this section, we will discuss *solving inequalities*.

The **inequality symbols** are:

- less than: $<$
- less than or equal to: $\leq$
- greater than: $>$
- greater than or equal to: $\geq$

Of course, when we write $x < 5$ we mean "all the values of x that are less than 5" and when we write $x \geq -2$ we mean "all the values of x that are greater than or equal to -2".

In this section, we will use these ideas, but we will also use another idea. Very often, it will be helpful to *interpret* expressions like $f(x) < 0$ or $f(x) \geq 0$, where f is some function of x. This is usually done in the graphical sense:

- When we write $f(x) < 0$, we are asking "for what values of x is the graph of f **strictly below** the x-axis?"
- When we write $f(x) \leq 0$, we are asking "for what values of x is the graph of f **at or below** the x-axis?"
- When we write $f(x) > 0$, we are asking "for what values of x is the graph of f **strictly above** the x-axis?"
- When we write $f(x) \geq 0$, we are asking "for what values of x is the graph of f **at or above** the x-axis?"

Note: When multiplying or dividing both sides of an inequality by a negative number, be sure to **change the sense of the inequality**; that is, reverse the inequality symbol.

Linear Polynomial Inequalities

Example 1. Solve: $3x + 5 < 7$.

Solution:

$$\begin{aligned} 3x + 5 &< 7 \\ 3x &< 2 \\ x &< \frac{2}{3} \end{aligned}$$

The solution is

$$\left\{ x \mid x < \frac{2}{3} \right\} = \left(-\infty, \frac{2}{3} \right).$$

Check your work using the "Insert relation" dialog in Graph.

1. Click on the button that looks like [$x < y$] (or hit [F6]) to open the "Insert relation" dialog.
2. On the "Relation" line, enter $3x + 5 < 7$.
3. From the "Style" drop-down menu, chose an appropriate style, like diagonal lines.
4. From the "Color" drop-down menu, choose an appropriate color, like green.
5. Make sure that "Width" is set to 0.
6. Select [OK] or hit [Enter].

Example 2. Solve: $-2x + 9 \le -5$.

Solution:

$$\begin{aligned} -2x + 9 &\le -5 \\ -2x &\le -14 \\ x &\ge 7 && \text{(reverse the inequality)} \end{aligned}$$

The solution is

$$\{x \,|\, x \ge 7\} = \big[7, \infty\big).$$

Check your work using the "Insert relation" dialog in Graph.

Non-Linear Polynomial Inequalities

The basic steps for solving a *non-linear polynomial inequality* are:

1. Move everything over to the left-hand side (LHS) so that 0 is on the right-hand side (RHS).
2. If the leading coefficient is negative, divide everything by -1 and reverse the inequality.
3. Interpret the meaning of the polynomial inequality in a graphical sense.
4. Solve the related polynomial equation–review **Sections 3.3** and **3.4**. Keep the real solutions and discard the complex solutions.
5. Use the real solutions of the related polynomial equation to divide the x-axis into intervals.
6. Pick a test value in each interval.
7. Evaluate the related polynomial at each test value (the input) to determine whether the polynomial's sign (the output) is **positive** or **negative**. If the sign is **positive**, then the graph of the polynomial is **above** the x-axis in that interval, and if the sign is **negative**, the the graph is **below** the x-axis in that interval.
8. Determine the intervals for which the original polynomial inequality is satisfied using the graphical interpretation from Step (3).

9. Include the endpoints of any solution intervals if $\leq$ or $\geq$; that is, use brackets appropriately. Do not include the endpoints of any solution intervals if $<$ or $>$; that is, use parentheses appropriately. Of course, never include $\pm\infty$.

10. Use Graph to verify your solutions. Simply graph the original polynomial inequality using the "Insert relation" dialog and the related polynomial equation using the "Insert function" dialog and compare the two graphs.

Example 3. Solve: $-6x < x^2 + 5$.

Solution: (Steps 1 & 2) Rearranging gives

$$\begin{aligned} -6x &< x^2 + 5 && \text{(original inequality)} \\ -x^2 - 6x - 5 &< 0 && \text{(move everything to LHS)} \\ \underbrace{x^2 + 6x + 5}_{f(x)} &> 0 && \text{(divide by } -1) \end{aligned}$$

(Step 3) This expression is asking "for what values of x is the graph of

$$f(x) = x^2 + 6x + 5$$

strictly **above** the x-axis?"

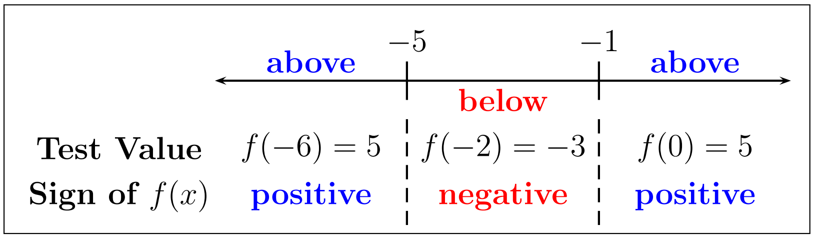
(Step 4) Setting $f(x) = 0$ gives

$$\begin{aligned} x^2 + 6x + 5 &= 0 && \text{("pretend" we have an equation)} \\ (x+5)(x+1) &= 0 && \text{(factor)} \end{aligned}$$

The solutions to the related equation are $x = -5$ or $x = -1$.

(Steps 5-7) Dividing the x-axis into intervals based on these solutions, picking a test value in each interval, and evaluating $f(x)$ at each of these test values gives the chart on the next page.

When the sign of f is **positive** at a test value in a particular interval, then the graph of f is **above** the x-axis in that interval. Likewise, when the sign of f is **negative** at a test value in a particular interval, then the graph of f



Test Value	$f(-6) = 5$	$f(-2) = -3$	$f(0) = 5$
Sign of $f(x)$	**positive**	**negative**	**positive**

is **below** the x-axis in that interval. Also, always try to pick $x = 0$ as a test point since $f(0)$ is usually easy to compute!

(Steps 8 & 9) Since we are looking for the values of x for which the graph of

$$f(x) = x^2 + 6x + 5$$

is **strictly** **above** the x-axis, do not include the endpoints of any solution intervals (we had $>$), so that the solution to the original polynomial inequality is

$$\big(-\infty, -5\big) \cup \big(-1, \infty\big).$$

(Step 10) Check your work using Graph.

Example 4. Solve: $x^5 \leq x^3$.

Solution: Rearranging gives

$$\underbrace{x^5 - x^3}_{f(x)} \leq 0 \qquad \text{(move everything to the LHS)}$$

This expression is asking "for what values of x is the graph of

$$f(x) = x^5 - x^3$$

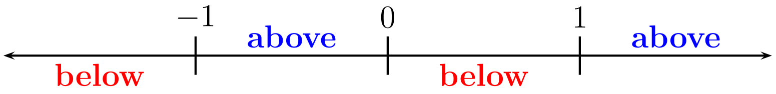
at or **below** the x-axis?"

Setting $f(x) = 0$ gives

$$\begin{aligned} x^5 - x^3 &= 0 && \text{(``pretend'' we have an equation)} \\ x^3(x^2 - 1) &= 0 && \text{(factor the GCF)} \\ x^3(x+1)(x-1) &= 0 && \text{(factor)} \end{aligned}$$

The solutions to the related equation are $x = -1$, $x = 0$, or $x = 1$.

Dividing the x-axis into intervals based on these solutions, picking a test value in each interval, and evaluating $f(x)$ at each of these test values gives



Test Value	$f(-2) = -24$	$f(-\frac{1}{2}) \approx 0.1$	$f(\frac{1}{2}) \approx -0.1$	$f(2) = 24$
Sign of $f(x)$	**negative**	**positive**	**negative**	**positive**

Since we are looking for the values of x for which the graph of

$$f(x) = x^5 - x^3$$

is **at or** below the x-axis, include the endpoints of any solution intervals (we had $\leq$), so that the solution to the original polynomial inequality is

$$\big(-\infty, -1\big] \cup \big[0, 1\big].$$

Check the solution in Graph.

Example 5. Solve: $x^5 + x^2 \geq 2x^3 + 2$.

Solution: Rearranging gives

$$\underbrace{x^5 - 2x^3 + x^2 - 2}_{f(x)} \geq 0 \qquad \text{(move everything to the LHS)}$$

This expression is asking "for what values of x is the graph of

$$f(x) = x^5 - 2x^3 + x^2 - 2$$

at or above the x-axis?"

Setting $f(x) = 0$ gives

$$\begin{aligned} x^5 - 2x^3 + x^2 - 2 &= 0 && \text{("pretend" we have an equation)} \\ (x^5 - 2x^3) + (x^2 - 2) &= 0 && \text{(factor by grouping)} \\ x^3(x^2 - 2) + (x^2 - 2) &= 0 && \text{(factor)} \\ (x^2 - 2)(x^3 + 1) &= 0 && \text{(factor the GCF)} \end{aligned}$$

Setting the first factor equal to 0 gives

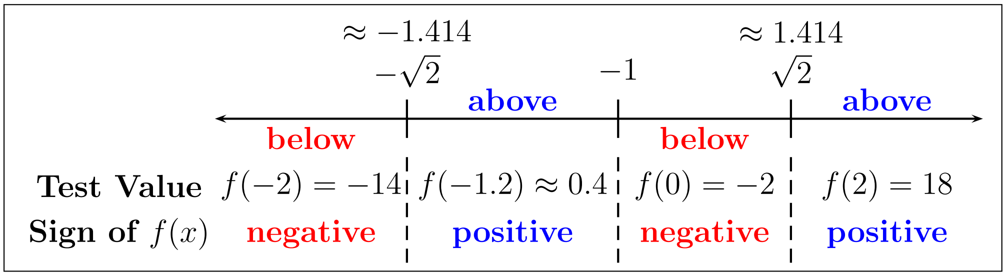
$$\begin{aligned} x^2 - 2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

with solutions $x = -\sqrt{2}$ or $x = \sqrt{2}$. Setting the second factor equal to 0 gives

$$\begin{aligned} x^3 + 1 &= 0 && \text{(sum of perfect cubes)} \\ (x+1)(x^2 - x + 1) &= 0 && \text{(review page 28)} \end{aligned}$$

with solutions $x = -1$, $x = \dfrac{1 + i\sqrt{3}}{2}$, or $x = \dfrac{1 - i\sqrt{3}}{2}$. After discarding the complex solutions, we are left with the real solutions to the related equation: $x = -\sqrt{2} \approx -1.414$, $x = -1$, or $x = \sqrt{2} \approx 1.414$.

Dividing the x-axis into intervals based on these real solutions, picking a test value in each interval, and evaluating $f(x)$ at each of these test values gives



	below	above	below	above
Test Value	$f(-2) = -14$	$f(-1.2) \approx 0.4$	$f(0) = -2$	$f(2) = 18$
Sign of $f(x)$	**negative**	**positive**	**negative**	**positive**

Since we are looking for the values of x for which the graph of

$$f(x) = x^5 - 2x^3 + x^2 - 2$$

is **at or above** the x-axis, include the endpoints of any solution intervals (we had $\geq$), so that the solution to the original polynomial inequality is

$$\big[-\sqrt{2}, -1\big] \cup \big[\sqrt{2}, \infty\big).$$

Check the solution in Graph.

Example 6. Solve: $x^2 + 12 < 4x$.

Solution: Rearranging gives

$$\underbrace{x^2 - 4x + 12}_{f(x)} < 0 \qquad\qquad \text{(move everything to the LHS)}$$

This expression is asking "for what values of x is the graph of

$$f(x) = x^2 - 4x + 12$$

strictly below the x-axis?"

Setting $f(x) = 0$ gives

$$x^2 - 4x + 12 = 0.$$

This does not factor "nicely", so use the quadratic formula to find the values of x that satisfy this equation: they are

$$\begin{aligned} x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(12)}}{2(1)} = \frac{4 \pm \sqrt{-32}}{2} &= \frac{4 \pm 4i\sqrt{2}}{2} \\ &= 2 \pm 2i\sqrt{2}. \end{aligned}$$

These values are complex conjugates so there are no real solutions to the related equation. Therefore, the graph of $f(x) = x^2 - 4x + 12$ is *never* **strictly below** the x-axis; that is, there is no solution to the original polynomial inequality. Thus, the solution may be expressed as

$$\text{no solution} = \{\ \} = \emptyset.$$

Check the solution in Graph.

Example 7. Solve: $x^2 + 12 > 4x$.

Solution: This is very similar to the polynomial inequality in Example 6 except the sense of the inequality has been reversed. Rearranging gives

$$\underbrace{x^2 - 4x + 12}_{f(x)} > 0 \qquad\qquad \text{(move everything to the LHS)}$$

This expression is asking "for what values of x is the graph of

$$f(x) = x^2 - 4x + 12$$

strictly above the x-axis?" Based on the solution to EXAMPLE 6, it should be clear that the graph of $f(x) = x^2 - 4x + 12$ is *always* **strictly above** the x-axis. Thus, the solution may be expressed as

$$\text{all reals} = \mathbb{R} = \big(-\infty, \infty\big).$$

Check the solution in Graph.

Rational Inequalities

The basic steps for solving a *rational inequality* are:

1. Move everything over to the left-hand side (LHS) so that 0 is on the right-hand side (RHS).
2. Interpret the meaning of the rational inequality in a graphical sense.
3. Find the values for which the rational function is not defined. Keep the real values that make any denominator 0, but discard the complex ones.
4. Solve the related rational equation–review **Section 2.5**. Keep the real solutions and discard the complex solutions.
5. The real values found in Step (3) and Step (4) are called **critical values**.
6. Use the critical values to divide the x-axis into intervals.
7. Pick a test value in each interval.
8. Evaluate the related rational function at each test value (the input) to determine whether the function's sign (the output) is **positive** or **negative**. If the sign is **positive**, then the graph of the rational function is **above** the x-axis in that interval, and if the sign is **negative**, the the graph is **below** the x-axis in that interval.
9. Determine the intervals for which the original rational inequality is satisfied using the graphical interpretation from Step (2).

10. If $\leq$ or $\geq$, include the critical values from Step (4); that is, use brackets appropriately. If $<$ or $>$, do not include the critical values from Step (4); that is, use parentheses appropriately. Of course, never include the critical values from Step (3), since these critical values are not in the domain of the rational function, and never include $\pm\infty$.
11. Use Graph to verify your solutions. Simply graph the original rational inequality using the "Insert relation" dialog and the related rational equation using the "Insert function" dialog.

Example 8. Solve: $\dfrac{-2}{5-x} \geq 0$.

Solution: In this example, everything is already on the LHS. This rational inequality is asking "for what values of x is the graph of

$$f(x) = \frac{-2}{5-x}$$

at or above the x-axis?"

The denominator of f is 0 when

$$5 - x = 0 \quad \Longrightarrow \quad x = 5,$$

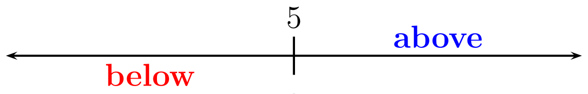
so $x = 5$ is a critical value.

Setting $f(x) = 0$ and multiplying by the LCD gives

$$\begin{aligned} \cancel{(5-x)}\frac{-2}{\cancel{5-x}} &= 0 \\ -2 &= 0, \end{aligned}$$

which is *nonsense*–this is *never* true! In other words, there are no other critical values, only $x = 5$.

Dividing the x-axis into intervals based on this critical value, picking a test value in each interval, and evaluating $f(x)$ at each of these test values gives the chart on the next page.



Test Value	$f(0) = -\frac{2}{5}$	$f(6) = 2$
Sign of $f(x)$	**negative**	**positive**

We are looking for the values of x for which the graph of

$$f(x) = \frac{-2}{5-x}$$

is **at or above** the x-axis, but since the critical value $x = 5$ makes the denominator 0, we will not include this critical value in the solution interval. Therefore, the solution to the original rational inequality is

$$(5, \infty).$$

Check the solution in Graph. Note that the graph of

$$f(x) = \frac{-2}{5-x}$$

never actually touches or crosses the x-axis! The graph has a vertical asymptote at the line $x = 5$ and the graph is always below the x-axis when $x < 5$ and always above the x-axis when $x > 5$.

Example 9. Solve: $\frac{x+5}{x-4} > \frac{3x+2}{2x+1}$.

Solution: Move everything over to the LHS to get

$$\underbrace{\frac{x+5}{x-4} - \frac{3x+2}{2x+1}}_{f(x)} > 0.$$

This expression is asking "for what values of x is the graph of

$$f(x) = \frac{x+5}{x-4} - \frac{3x+2}{2x+1}$$

strictly above the x-axis?"

The denominator is zero when

$$x = 4 \quad \text{or} \quad x = -\frac{1}{2},$$

which gives two critical values for f.

Setting $f(x) = 0$ and solving for x gives (multiply by the LCD)

$$\begin{aligned}
\cancel{(x-4)}(2x+1)\left[\frac{x+5}{\cancel{x-4}}\right] - (x-4)\cancel{(2x+1)}\left[\frac{3x+2}{\cancel{2x+1}}\right] &= 0 \\
(2x+1)(x+5) - (x-4)(3x+2) &= 0 \\
[2x^2 + 11x + 5] - [3x^2 - 10x - 8] &= 0 \\
2x^2 + 11x + 5 - 3x^2 + 10x + 8 &= 0 \\
-x^2 + 21x + 13 &= 0.
\end{aligned}$$

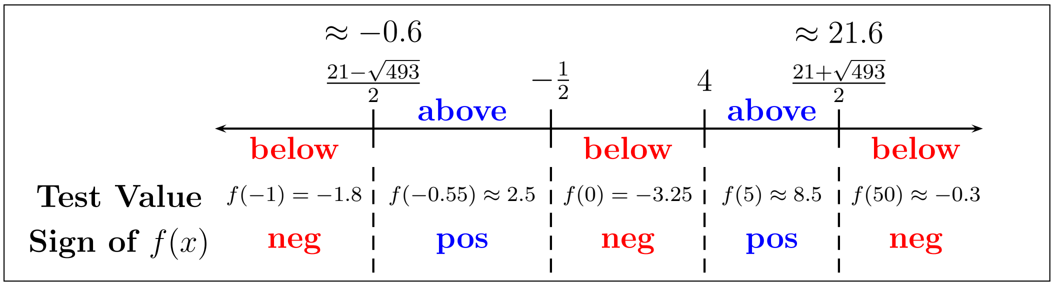
Using the **Quadratic Formula** to solve this quadratic equation yields

$$\begin{aligned}
x = \frac{-21 \pm \sqrt{(21)^2 - 4(-1)(13)}}{2(-1)} &= \frac{-21 \pm \sqrt{441+52}}{-2} \\
&= \frac{21 \pm \sqrt{493}}{2},
\end{aligned}$$

which gives two more critical values for f, namely,

$$x = \frac{21 - \sqrt{493}}{2} \approx -0.6 \quad \text{or} \quad x = \frac{21 + \sqrt{493}}{2} \approx 21.6.$$

Dividing the x-axis into intervals based on these critical values, picking a test value in each interval, and evaluating $f(x)$ at each of these test values gives



We are looking for the values of x for which the graph of

$$f(x) = \frac{x+5}{x-4} - \frac{3x+2}{2x+1}$$

is **strictly above** the x-axis, so do not include any of the critical values in the solution interval. Thus, the solution to the original rational inequality is

$$\left(\frac{21-\sqrt{493}}{2}, -\frac{1}{2}\right) \cup \left(4, \frac{21+\sqrt{493}}{2}\right).$$

Check the solution in Graph.

Example 10. Solve: $\dfrac{3}{x^2-4} \leq \dfrac{5}{x^2+7x+10}$.

Solution: Move everything over to the LHS to get

$$\underbrace{\frac{3}{x^2-4} - \frac{5}{x^2+7x+10}}_{f(x)} \leq 0.$$

This expression is asking "for what values of x is the graph of

$$\begin{aligned} f(x) &= \frac{3}{x^2-4} - \frac{5}{x^2+7x+10} \\ &= \frac{3}{(x+2)(x-2)} - \frac{5}{(x+2)(x+5)} \end{aligned}$$

at or below the x-axis?"

The denominator is zero when $x = -5$, $x = -2$, or $x = 2$, which gives three critical values for f.

Setting $f(x) = 0$ and solving for x gives (multiply by the LCD)

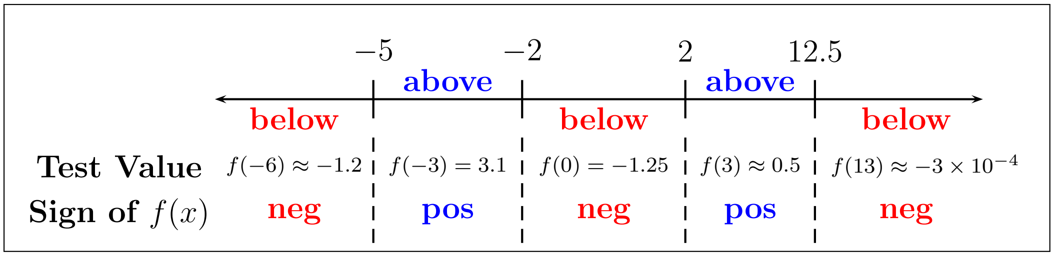
$$\cancel{(x+2)}\cancel{(x-2)}(x+5)\left[\frac{3}{\cancel{(x+2)}\cancel{(x-2)}}\right] - \cancel{(x+2)}(x-2)\cancel{(x+5)}\left[\frac{5}{\cancel{(x+2)}\cancel{(x+5)}}\right] = 0$$

which simplifies as

$$\begin{aligned} 3(x+5) - 5(x-2) &= 0 \\ 3x + 15 - 5x + 10 &= 0 \\ -2x + 25 &= 0 \\ -2x &= -25 \\ x &= \frac{25}{2}. \end{aligned}$$

This gives another critical value for f: $x = 25/2 = 12.5$.

Dividing the x-axis into intervals based on these critical values, picking a test value in each interval, and evaluating $f(x)$ at each of these test values gives



	below	above	below	above	below
Test Value	$f(-6) \approx -1.2$	$f(-3) = 3.1$	$f(0) = -1.25$	$f(3) \approx 0.5$	$f(13) \approx -3 \times 10^{-4}$
Sign of $f(x)$	**neg**	**pos**	**neg**	**pos**	**neg**

We are looking for the values of x for which the graph of

$$\begin{aligned} f(x) &= \frac{3}{x^2 - 4} - \frac{5}{x^2 + 7x + 10} \\ &= \frac{3}{(x+2)(x-2)} - \frac{5}{(x+2)(x+5)} \end{aligned}$$

is **at or** below the x-axis, so include the critical value $x = 12.5$ in the solution. Do not include the critical values $x = -5$, $x = -2$, or $x = 2$ since they make the denominator zero. Thus, the solution to the original rational inequality is

$$\big(-\infty, -5\big) \cup \big(-2, 2\big) \cup \big[12.5, \infty\big).$$

Check the solution in Graph.