Section 3.5: Rational Functions

Key points:

- Find the domain of a rational function.
- Determine whether the graph of a rational function has holes or vertical asymptotes.
- Find the holes or vertical asymptotes, if there are any.
- Determine whether the graph of a rational function has a horizontal asymptote or a slant/oblique asymptote .
- Find the horizontal asymptote or slant asymptote, if there is one.
- Find the zeros of a rational function. The real zeros give information about the x-intercepts for the graph.
- Find the y-intercept of a rational function.
- Solve applied problems involving rational functions.

Rational Function

A rational function f is the quotient of two polynomial functions p and q; that is,

$$f(x) = \frac{p(x)}{q(x)},$$

where $q(x)$ is not the zero polynomial. The polynomial $p(x)$ is the *numerator* of f and the polynomial $q(x)$ is the *denominator* of f.

Domain of a Rational Function

The domain of a rational function $f(x) = p(x)/q(x)$ is the set of all real numbers x such that $q(x) \neq 0$. Every real root of the denominator q must be excluded from the domain.

Example 1. The domain of

$$f(x) = \frac{x+3}{x^2+5x+4} = \frac{x+3}{(x+4)(x+1)}$$

is the set of all real numbers except -4 and -1; the domain is

$$\{x|x \neq -4, -1\} = (-\infty, -4) \cup (-4, -1) \cup (-1, \infty).$$

Example 2. The domain of

$$g(x) = \frac{x+7}{5x^2+9}$$

is **all real numbers** $= \mathbb{R} = (-\infty, \infty)$. The only values of x that make the denominator zero are

$$\begin{aligned} 5x^2 + 9 &= 0 \\ x^2 &= \frac{-9}{5} \\ x &= \pm\frac{3i}{\sqrt{5}} = \pm\frac{3i\sqrt{5}}{5}, \end{aligned}$$

which are complex.

Holes

Suppose f is a rational function $f(x) = p(x)/q(x)$ where the numerator and denominator have *common factors*. After the domain of f has been found, the rational function can be *reduced* by canceling the common factors.

Example 3. The graphs of the rational functions f and g in Examples 1 and 2, respectively, have no holes since the numerator and denominator of each of these functions have no common factors.

Example 4. The rational function

$$h(x) = \frac{3x-6}{x^2-x-2} = \frac{3(x-2)}{(x-2)(x+1)} = \frac{3}{x+1}$$

has domain $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$. There is a hole in the graph of h when $x = 2$ since the numerator and denominator have a common factor $(x - 2)$.

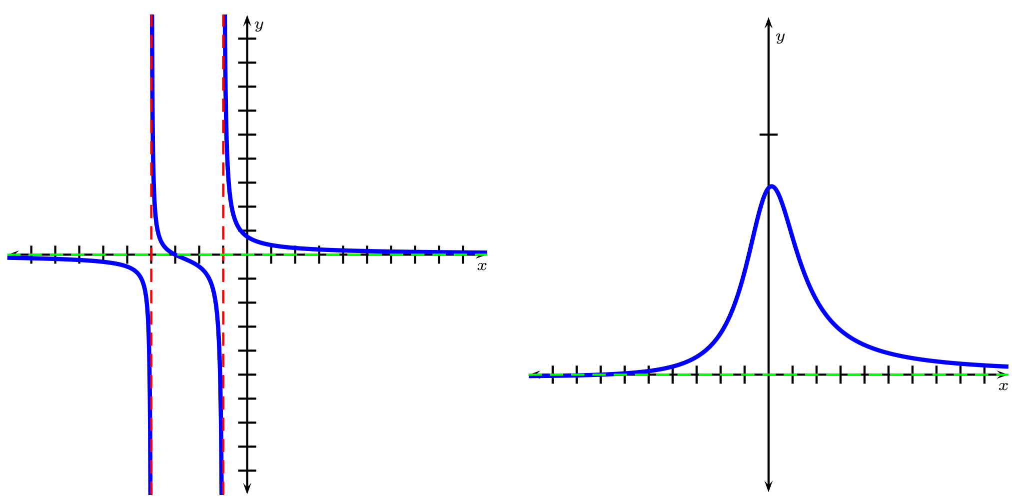
The hole is at the point $(2, 1)$. The y-coordinate for the hole in the graph of h was found by computing $h(2)$ using the reduced form of h.

Once holes have been determined for a rational function, we may search for vertical asymptotes.

Vertical Asymptotes

Suppose $f(x) = p(x)/q(x)$ is a rational function that has been reduced. If the real number a is a zero of the denominator, this is, if $q(a) = 0$, then the graph of f has a vertical asymptote at the **line** $x = a$.

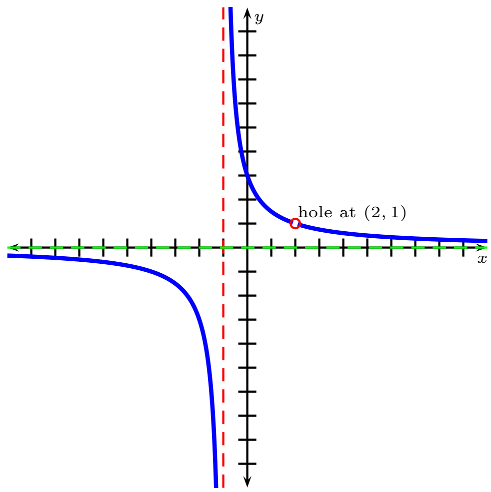
Example 5. The graph of the rational function f of EXAMPLE 1 has vertical asymptotes at $x = -4$ and $x = -1$. The graph of the rational function g of EXAMPLE 2 has no vertical asymptotes. See FIGURE 1.



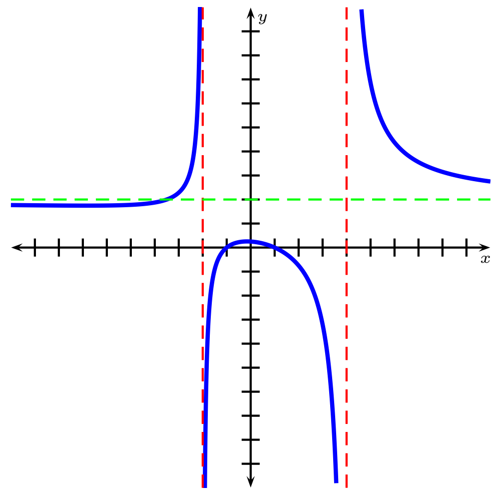
(a) $f(x) = \frac{x+3}{x^2+5x+4} = \frac{x+3}{(x+4)(x+1)}$ (b) $g(x) = \frac{x+7}{5x^2+9}$

Figure 1: The graph of f has vertical asymptotes at the lines $x = -4$ and $x = -1$. The graph of g has no vertical asymptotes.

Example 6. The graph of the rational function h of EXAMPLE 4 has a hole at the point $(2, 1)$ and a vertical asymptote at the line $x = -1$. See FIGURE 2(a).



(a) $h(x) = \frac{3x-6}{x^2-x-2} = \frac{3(x-2)}{(x-2)(x+1)} = \frac{3}{x+1}$, where $x \neq 2$



(b) $j(x) = \frac{2x^2-2}{x^2-2x-8} = \frac{2(x-1)(x+1)}{(x+2)(x-4)}$

Figure 2: The graph of h has a hole at $(2, 1)$ and a vertical asymptote at the line $x = -1$. The graph of j has 2 vertical asymptotes: the lines $x = -2$ and $x = 4$.

Example 7. The rational function

$$j(x) = \frac{2x^2 - 2}{x^2 - 2x - 8} = \frac{2(x-1)(x+1)}{(x+2)(x-4)}$$

has domain $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$. The graph of j has vertical asymptotes at the lines $x = -2$ and $x = 4$. See Figure 2(b).

Thus, if $x = a$ is a real root of the denominator of a rational function, then the graph will have either a hole when $x = a$ or a vertical asymptote at the line $x = a$. There will be a hole in the graph when the numerator and denominator have common factor $(x - a)$. There will be a vertical asymptote when the denominator has factor $(x - a)$ but the numerator does not.

It is possible for the graph of a rational function to have no holes or vertical asymptotes, as seen in Figure 1(b). It is possible for the graph of a rational function to have a hole and a vertical asymptote, as seen in Figure 2(a). In fact, a rational function *may* have numerous holes or numerous vertical asymptotes, or none at all!

Horizontal Asymptote

Suppose $f(x) = p(x)/q(x)$ is a rational function.

- If the degree of the numerator **is less than** the degree of the denominator, then the **line**

$$y = 0 \qquad \text{(the } x\text{-axis)}$$

  is the horizontal asymptote for the graph of f.

- If the degree of the numerator **equals** the degree of the denominator, then the **line**

$$y = \frac{\text{lead coef of num}}{\text{lead coef of denom}}$$

  is the horizontal asymptote for the graph of f.

- If the degree of the numerator **is greater than** the degree of the denominator, then the graph of f has no horizontal asymptote.

The graph of a rational function may have *at most* one horizontal asymptote.

Example 8. The graphs of the rational functions f, g, and h given previously each have an horizontal asymptote at the line $y = 0$ (the x-axis) since the degree of the numerator is less than the degree of the denominator for each of these functions. Refer to FIGURE 1 and FIGURE 2(a) for the graphs of these rational functions.

Example 9. The graph of the rational function j given previously has a horizontal asymptote at the line

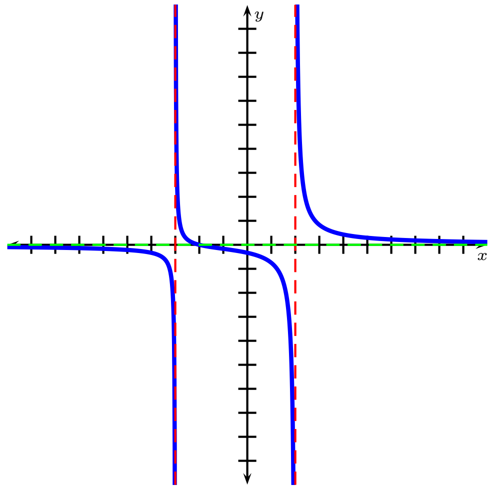
$$y = \frac{\text{lead coef of num}}{\text{lead coef of denom}} = \frac{2}{1} = 2,$$

since the degree of the numerator is equal to the degree of the denominator for this function. Refer to FIGURE 2(b).

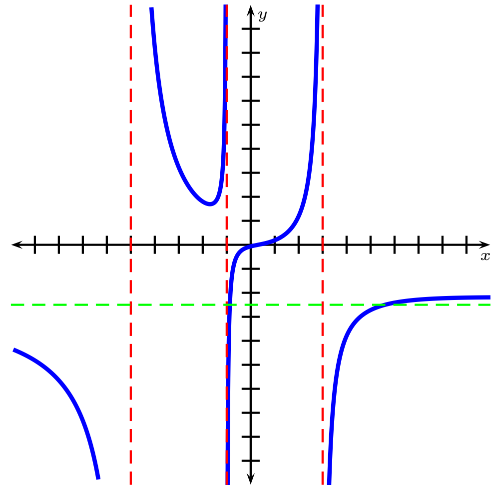
Example 10. The rational function

$$f(x) = \frac{x+2}{x^2+x-6} = \frac{x+2}{(x+3)(x-2)}$$

has a horizontal asymptote at the line $y = 0$ (the x-axis) since the degree of the numerator is less than the degree of the denominator. See FIGURE 3(a).



(a) $f(x) = \frac{x+2}{x^2+x-6}$



(b) $g(x) = \frac{-5x^3-6x+2}{2x^3+6x^2-26x-30}$

Figure 3: The graph of f has vertical asymptotes at the lines $x = -3$ and $x = 2$ and a horizontal asymptote at the line $y = 0$. The graph of g has vertical asymptotes at the lines $x = -5$, $x = -1$, and $x = 3$ and a horizontal asymptote at the line $y = -5/2$.

Example 11. The rational function

$$g(x) = \frac{-5x^3 - 6x + 2}{2x^3 + 6x^2 - 26x - 30} = \frac{-5x^3 - 6x + 2}{2(x+5)(x+1)(x-3)}$$

has a horizontal asymptote at the line $y = -\frac{5}{2}$ since the degree of the numerator is equal to the degree of the denominator. See FIGURE 3(b).

Example 12. The rational function

$$h(x) = \frac{-2x^5 + x^3 - 7x + 1}{3x^2 + x + 2}$$

does not have a horizontal asymptote since the degree of the numerator is greater than the degree of the denominator.

Remark: Notice that the graph of f and the graph of g in FIGURE 3 cross the horizontal asymptote. The graph of a rational function will **never** cross a vertical asymptote, but it **may** cross a horizontal asymptote.

Although there may be many holes or vertical asymptotes for the graph of a rational function, there is *at most* one horizontal asymptote.

Slant/Oblique Asymptote

Suppose $f(x) = p(x)/q(x)$ is a rational function. If the degree of the numerator **is exactly one more than** the degree of the denominator, then the graph of f will have a slant, or oblique, asymptote.

If a rational function has a horizontal asymptote, then there is no need to look for a slant asymptote–there cannot be one. The graph of a rational function may have neither a horizontal nor a slant asymptote, but it will never have both!

The equation of a slant asymptote, if there is one, will be

$$y = \text{“quotient”},$$

where "quotient" is found by dividing the numerator $p(x)$ by the denominator $q(x)$. In general, long-division must be used to find the slant asymptote. However, if the denominator is of the form $(x - c)$, then synthetic division may be used.

Example 13. The graph of the rational function h in EXAMPLE 12 does not have a vertical asymptote since the roots of the denominator are complex conjugates (check the discriminant); it does not have a horizontal asymptote since the degree of the numerator is greater than the degree of the denominator; and, it does not have a slant asymptote since the degree of the numerator is two more than the degree of the denominator.

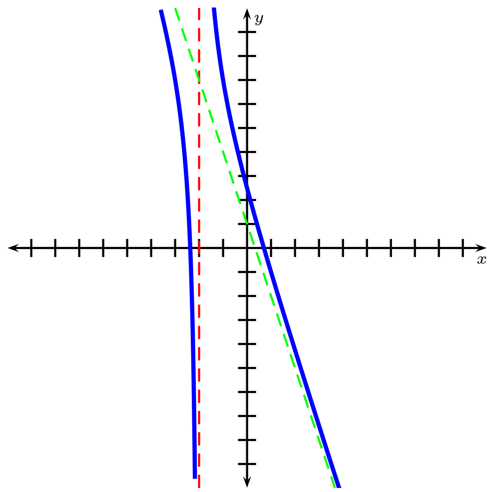
Example 14. The graph of

$$m(x) = \frac{-3x^2 - 5x + 5}{x + 2}$$

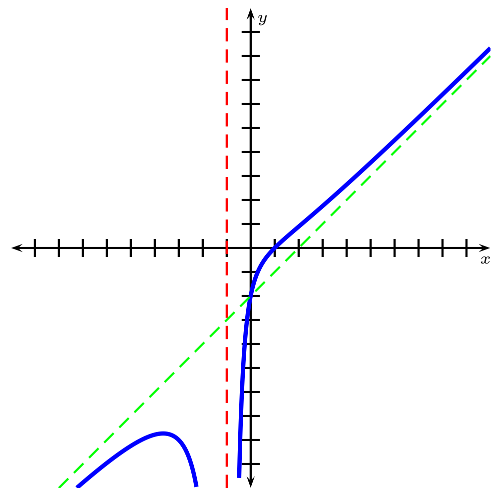
does not have a horizontal asymptote, but it does have a slant asymptote since the degree of the numerator is exactly one more that the degree of the denominator. Moreover, since the denominator is of the form $(x - (-2))$, synthetic division may be used to find the slant asymptote, as follows:

$$\begin{array}{r|rrr} \underline{-2} & -3 & -5 & 5 \\ & \downarrow & 6 & -2 \\ \hline & -3 & 1 & \underline{|3} \end{array}$$

Discarding the remainder gives the slant asymptote: it is the line $y = -3x + 1$. In addition, there is a vertical asymptote at the line $x = -2$. See FIGURE 4(a).



(a) $m(x) = \frac{-3x^2-5x+5}{x+2}$



(b) $n(x) = \frac{x^3+x-2}{x^2+2x+1}$

Figure 4: The graph of m has a vertical asymptote at the line $x = -2$ and a slant asymptote at the line $y = -3x + 1$. The graph of n has a vertical asymptote at the line $x = -1$ and a slant asymptote at the line $y = x - 2$.

Example 15. The graph of

$$n(x) = \frac{x^3 + x - 2}{x^2 + 2x + 1} = \frac{(x-1)(x^2+x+2)}{(x+1)^2}$$

does not have a horizontal asymptote, but it does have a slant asymptote since the degree of the numerator is exactly one more than the degree of the denominator. Long-division must be used to find it: it is the line $y = x - 2$. There is also a vertical asymptote at the line $x = -1$. See FIGURE 4(b).

Zeros of a Rational Function

Suppose $f(x) = p(x)/q(x)$ is a rational function and the numerator and denominator have no common factors. Then the zeros of f are the values of x for which the numerator is zero.

To find the zeros of f, simply set the numerator equal to zero and solve for x. The real roots of the numerator will be the x-intercepts for the graph of the rational function f.

Example 16. The zeros of the rational function m given in Example 14 can be found by setting the numerator equal to zero and solving for x; using the quadratic formula, the roots of $-3x^2 - 5x + 5 = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-3)(5)}}{2(-3)} = \frac{-5 \pm \sqrt{85}}{6}.$$

These two distinct real roots give two x-intercepts, the points

$$\left(\frac{-5 - \sqrt{85}}{6}, 0 \right) \approx (-2.37, 0)$$

and

$$\left(\frac{-5 + \sqrt{85}}{6}, 0 \right) \approx (0.7, 0).$$

Example 17. The zeros of the rational function n given in Example 15 can be found similarly. This function has one real root $x = 1$ and two complex conjugate roots, $x = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$. There is an x-intercept at the point $(1, 0)$.

The y-intercept of a Rational Function

Suppose $f(x) = p(x)/q(x)$ is a rational function and the numerator and denominator have no common factors. The y-intercept of f is found the same way as for any function: let $x = 0$ and compute $f(0)$; that is, the y=intercept is the point $\big(0, f(0)\big)$.

Example 18. The rational function m has y-intercept $(0, 5/2)$ and the rational function n has y-intercept $(0, -2)$.