Section 3.4: Theorems about Zeros of Polynomials

Key points:

- The Fundamental Theorem of Algebra: p. 290.
- Linear Factorization Theorem: p. 290.
- Finding polynomials with given zeros: p. 291.
- Non-real Zeros Theorem: p. 291.
- Irrational Zeros Theorem: p. 292.
- Rational Zeros Theorem: p. 293.
- Descartes' Rule of Signs: p. 296.

The Fundamental Theorem of Algebra: Every polynomial with degree ≥ 1 has a zero.

Linear Factorization Theorem: Every polynomial with degree ≥ 1 can be written as

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$

where a_n is the leading coefficient and c_1, c_2, $\ldots$, c_n are the zeros of f.

Remember, **roots and factors "go together"!**

Non-real Zeros Theorem: Suppose $f(x)$ is a polynomial with REAL coefficients. If $x = a + bi$ is a zero, then so is $x = a - bi$. For polynomials with REAL coefficients, non-real roots occur in complex conjugate pairs.

Example 1. Given that $x = 2i$ is a zero of $f(x) = x^3 + x^2 + 4x + 4$, we know that $x = -2i$ is also a zero of f. Thus, two factors of f are $(x - 2i)$ and $(x + 2i)$. The polynomial f can be factored as (factor by grouping!)

$$\begin{aligned} f(x) &= x^3 + x^2 + 4x + 4 \\ &= (x + 1)(x - 2i)(x + 2i) \end{aligned}$$

This polynomial has only one real root $x = -1$, so it has only one x-intercept at $(-1, 0)$. The other two roots are complex conjugates.

Irrational Zeros Theorem: Suppose $f(x)$ is a polynomial with RATIONAL coefficients. If $x = a + b\sqrt{c}$ is a zero, then so is $x = a - b\sqrt{c}$. For polynomials with RATIONAL coefficients, irrational roots occur in conjugate pairs.

Example 2. Given that $x = -\sqrt{3}$ is a root of $g(x) = x^3 - 4x^2 - 3x + 12$, we know that $x = \sqrt{3}$ is also a root of g. Therefore, two factors of g are $(x + \sqrt{3})$ and $(x - \sqrt{3})$. Using synthetic division (or factor by grouping), g can be factored as

$$\begin{aligned} g(x) &= x^3 - 4x^2 - 3x + 12 \\ &= (x + \sqrt{3})(x - \sqrt{3})(x - 4) \end{aligned}$$

This polynomial has 3 real roots $x = \pm\sqrt{3}$ and $x = 4$, so it has three x-intercepts: $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$, and $(4, 0)$.

Rational Zeros Theorem: Suppose $f(x)$ is a polynomial with INTEGER coefficients and nonzero constant term. Then the only ***possible*** *rational zeros* are

$$\frac{\pm\{\text{factors of constant term}\}}{\pm\{\text{factors of lead. coef.}\}}$$

Example 3. If the polynomial $h(x) = 3x^5 + x^2 - 6$ has rational zeros, then they will appear in the following list:

$$\begin{aligned} \frac{\pm\{\text{factors of constant term}\}}{\pm\{\text{factors of lead. coef.}\}} &= \frac{\pm\{1, 2, 3, 6\}}{\pm\{1, 3\}} \\ &= \pm\{1, 2, 3, 6, \frac{1}{3}, \frac{2}{3}\} \end{aligned}$$

Use synthetic division to check if any of these *possible* rational zeros are *actual* zeros.

Although we probably will not use it too much, Descartes' Rule of Signs may be helpful in narrowing down the search for the real roots of a polynomial.

Descartes' Rule of Signs: Suppose $f(x)$ is a polynomial with REAL coefficients and nonzero constant term.

- **Positive Real Zeros:** Count the number of sign variations in $f(x)$. The number of positive real zeros = the number of sign variations in $f(x)$ **OR** differs from that by an even integer.

- **Negative Real Zeros:** Compute $f(-x)$ and count the number of sign variations in $f(-x)$. The number of negative real zeros = the number of sign variations in $f(-x)$ **OR** differs from that by an even integer.

Example 4. The polynomial $m(x) = x^3 + 3x^2 + 4x + 6$ has no sign variations. This means that there are no positive real roots. Replacing x with $-x$ gives

$$\begin{aligned} m(-x) &= (-x)^3 + 3(-x)^2 + 4(-x) + 6 \\ &= -x^3 + 3x^2 - 4x + 6, \end{aligned}$$

which has 3 sign variations. Thus, m either has 3 negative real roots or 1 negative real root and 2 complex conjugate roots. We are assured that m has *at least* one x-intercept!

It is probably much easier to simply graph a function and see how many positive and negative real roots it has!