

Section 2.2: Complex Numbers

Key point:

- Perform computations involving complex numbers.

The Imaginary Unit

The number i : The imaginary unit is the number i defined by

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}.$$

This definition allows us to take the square root of a negative number. However, when we do this, we end up with a complex number (imaginary number), not a real number.

Example 1. Simplify: (a) $\sqrt{-49}$ and (b) $\sqrt{-250}$.

$$\begin{aligned} \text{(a) } \sqrt{-49} &= \sqrt{49} \cdot \sqrt{-1} & \text{(b) } \sqrt{-250} &= \sqrt{250} \cdot \sqrt{-1} \\ &= 7i & &= \sqrt{25} \cdot \sqrt{10} \cdot \sqrt{-1} \\ & & &= 5i\sqrt{10} \end{aligned}$$

Example 2. Simplify: $\sqrt{-\frac{48}{7}}$.

$$\begin{aligned} \sqrt{-\frac{48}{7}} &= \sqrt{\frac{48}{7}} \cdot \sqrt{-1} = \frac{\sqrt{48}}{\sqrt{7}} \cdot \sqrt{-1} = \frac{\sqrt{16}\sqrt{3}}{\sqrt{7}} \cdot \sqrt{-1} = \frac{4i\sqrt{3}}{\sqrt{7}} \\ &= \frac{4i\sqrt{3}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{4i\sqrt{21}}{7}, \end{aligned}$$

where in the next to the last step, we do something called *rationalizing the denominator*. Very often, when we have square roots in the denominator, we try to “make them go away” by rationalizing the denominator.

Complex Numbers

Complex number: A complex number is a number of the form

$$a + bi,$$

where a and b are real numbers and i is the imaginary unit. The number a is called the *real part* of the complex number and the number b is called the *imaginary part* of the complex number.

Complex conjugate: The complex conjugate of $a + bi$ is the complex number $a - bi$.

Example 3. The following are some complex numbers and their conjugates:

complex number	complex conjugate
$3 + 2i$	$3 - 2i$
$-\frac{1}{3}i$	$\frac{1}{3}i$
$\frac{1}{2} - \frac{\sqrt{3}}{2}i$	$\frac{1}{2} + \frac{\sqrt{3}}{2}i$
14	14

Note that any real number, like 14, can be written as $14 + 0i$, and therefore the set real numbers is a subset (or is contained in) the set of complex numbers. See pages 2 and 195 for more information about number systems.

Example 4. Add: $(3 - 4i) + (-5 + 7i)$.

$$(3 - 4i) + (-5 + 7i) = -2 + 3i.$$

Just combine like terms!

Example 5. Subtract: $(-2 + i) - (4 - 3i)$

$$(-2 + i) - (4 - 3i) = -2 + i - 4 + 3i = -6 + 4i.$$

Distribute the negative and combine like terms.

Example 6. Multiply: $(2 + 3i)(4 - i)$.

$$(2 + 3i)(4 - i) = 8 - 2i + 12i - 3i^2 = 8 + 10i - 3(-1) = 11 + 10i.$$

Use the FOIL method to multiply the expressions and use the fact that $i^2 = -1$.

Example 7. Divide: $\frac{3 + i}{5 - 2i}$.

In order to divide two complex numbers, we need to *rationalize the denominator* by **multiplying top and bottom by the conjugate of the denominator**:

$$\begin{aligned}\frac{3 + i}{5 - 2i} &= \frac{3 + i}{5 - 2i} \cdot \frac{5 + 2i}{5 + 2i} = \frac{15 + 6i + 5i + 2i^2}{25 + 10i - 10i - 4i^2} \\ &= \frac{15 + 11i + 2(-1)}{25 - 4(-1)} \\ &= \frac{13 + 11i}{29} = \frac{13}{29} + \frac{11}{29}i.\end{aligned}$$

After using the FOIL method, we simplify the expression by combining like terms and replacing i^2 with -1 . Always write the final answer in the form $a + bi$!

Example 8. Simplify: (a) i^{50} and (b) i^{-97} .

Here, we use a “trick” to make the computations easier: we try to re-write the expressions in terms of i^2 , which we will then replace with -1 !

$$(a) \ i^{50} = (i^2)^{25} = (-1)^{25} = -1$$

$$\begin{aligned}(b) \ i^{-97} &= \frac{1}{i^{97}} = \frac{1}{i^{96} \cdot i} = \frac{1}{(i^2)^{48} \cdot i} = \frac{1}{(-1)^{48} \cdot i} = \frac{1}{i} \\ &= \frac{1}{i} \cdot \frac{-i}{-i} \\ &= \frac{-i}{-i^2} \\ &= \frac{-i}{-(-1)} = -i\end{aligned}$$

Notice in Part (b) that we had to rationalize the denominator!