Section 1.7: Symmetry and Transformations

Key points:

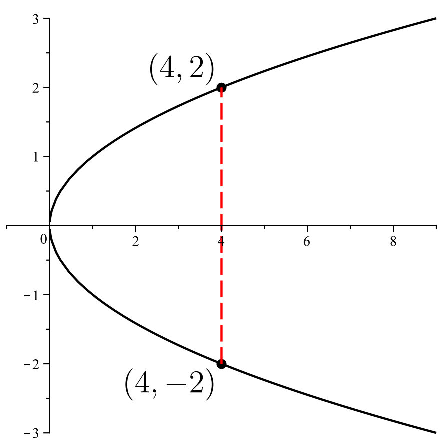
- Determine whether a graph is symmetric with respect to the x-axis, the y-axis, and the origin.
- Determine whether a function is even, odd, or neither.
- Recognize the names, equations, and graphs of the "seven basic functions".
- Given the equation or graph of a function, describe and graph various transformations of that function.

Symmetry

Symmetry with respect to the x-axis

For every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.

Example 1. The graph of $x = y^2$ is symmetric w/r/t the x-axis.



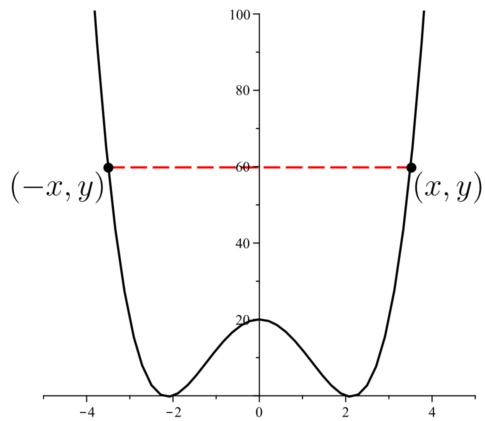
Notice that we can draw a *vertical line segment* between the point $(4, 2)$ and the point $(4, -2)$. In fact, if (x, y) is any point on the graph, a vertical line can be drawn to the point $(x, -y)$ on the graph since the graph of $x = y^2$ is symmetric w/r/t the x-axis.

Notice also that if we fold the graph of $x = y^2$ along the x-axis, the two halves match-up. Any graph that can be folded along the x-axis and the halves match-up will be symmetric w/r/t the x-axis. This provides us with a quick way to determine visually if an equation is symmetric w/r/t the x-axis. All we have to do is graph the equation and see if it can be folded over the x-axis!

Symmetry with respect to the y-axis

For every point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

Example 2. The graph of $y = x^4 - 9x^2 + 20$ is symmetric w/r/t the y-axis.



If (x, y) is any point on the graph, a *horizontal line segment* can be drawn to the point $(-x, y)$ on the graph since the graph of

$$y = x^4 - 9x^2 + 20$$

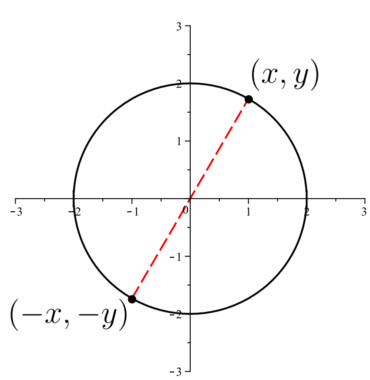
is symmetric w/r/t the y-axis.

If we fold the graph of $y = x^4 - 9x^2 + 20$ along the y-axis, the two halves match-up. Any graph that can be folded along the y-axis and the halves match-up will be symmetric w/r/t the y-axis. This provides us with a quick way to determine visually if an equation is symmetric w/r/t the y-axis. All we have to do is graph the equation and see if it can be folded over the y-axis!

Symmetry with respect to the origin

For every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.

Example 3. The graph of $x^2 + y^2 = 4$ is symmetric w/r/t the origin.



If (x, y) is any point on the graph, a line segment can be drawn *through the origin* to the point $(-x, -y)$ on the graph since the graph of $x^2 + y^2 = 4$ is symmetric w/r/t the origin.

Notice that this graph is also symmetric w/r/t the x-axis and the y-axis.

If we rotate the graph of $x^2 + y^2 = 4$ through 180°, we end up with the same graph. Any graph that can be rotated 180° and we end up with the same graph is symmetric w/r/t the origin. This provides us with a quick way to determine visually if an equation is symmetric w/r/t the origin. All we have to do is graph the equation and see if a 180° rotation gives us the original graph back!

Even and Odd Functions

Even: If $f(-x) = f(x)$, then plugging in the opposite of an x value gives us the same y value. This means that f is symmetric w/r/t the y-axis and we say that f is an **even function**. Every function that is symmetric w/r/t the y-axis is even, and every even function is symmetric w/r/t the y-axis.

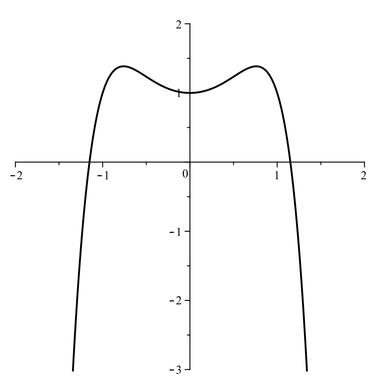
Odd: If $f(-x) = -f(x)$, then plugging in the opposite of an x value gives us the opposite of the y value. This means that f is symmetric w/r/t the origin and we say that f is an **odd function**. Every function that is symmetric w/r/t the origin is odd, and every odd function is symmetric w/r/t the origin.

Note 1: We only use the terms "even" and "odd" in reference to functions. If an equation is not the equation of a function, then these two terms do not apply.

Note 2: There is only one function that is *both even and odd*, the constant linear function $f(x) = 0$. It is also possible for a function to be **neither even nor odd**.

Example 4. Describe the following functions as even, odd, or neither:

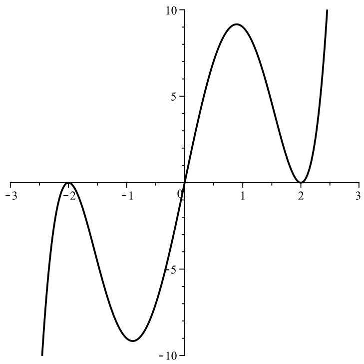
(a) $f(x) = -x^6 + x^2 + 1$.



$$\begin{aligned} f(-x) &= -(-x)^6 + (-x)^2 + 1 \\ &= -x^6 + x^2 + 1 \\ &= f(x), \end{aligned}$$

Since $f(-x) = f(x)$, the function f is **even**. Looking at the graph, we can easily verify that f is even because its graph is symmetric w/r/t the y-axis. Notice too that all of the exponents of f are even numbers.

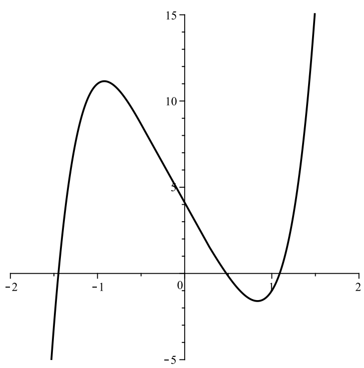
(b) $g(x) = x^5 - 8x^3 + 16x$.



$$\begin{aligned} g(-x) &= (-x)^5 - 8(-x)^3 + 16(-x) \\ &= -x^5 + 8x^3 - 16x \\ &= -(x^5 - 8x^3 + 16x) \\ &= -g(x) \end{aligned}$$

Since $g(-x) = -g(x)$, the function g is **odd**. Looking at the graph, we can easily verify that g is odd because its graph is symmetric w/r/t the origin. Notice too that all of the exponents of g are odd numbers.

(c) $h(x) = 3x^5 + x^2 - 9x + 4$.



$$\begin{aligned} h(-x) &= 3(-x)^5 + (-x)^2 - 9(-x) + 4 \\ &= -3x^5 + x^2 + 9x + 4 \\ &\neq h(x) \\ &\neq -h(x) \end{aligned}$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, the function h is **neither even nor odd**. Looking at the graph, we can easily verify that h is *not even* because its graph is not symmetric w/r/t the y-axis and h is *not odd* because its graph is not symmetric w/r/t the origin. Notice too that the exponents of h are a mix of even and odd numbers.

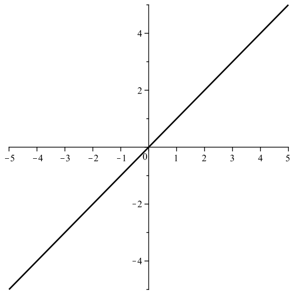
The Seven Basic Functions

As you continue your study of algebra, you will start to recognize certain ideas and concepts as *useful*. Being able to quickly sketch the graphs of the following seven functions can be extremely useful in this course, especially when we relate these graphs to the idea of transformations.

Please study the names, equations, and graphs of these functions until you feel comfortable with them. These functions serve as the building blocks for many other functions.

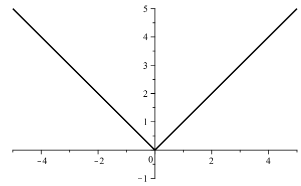
Identity function

$$f(x) = x$$



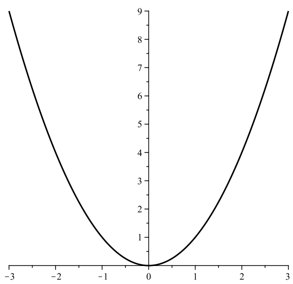
Absolute value function

$$f(x) = |x|$$



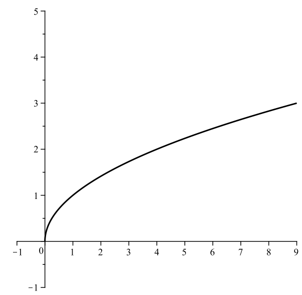
Squaring function

$$f(x) = x^2$$



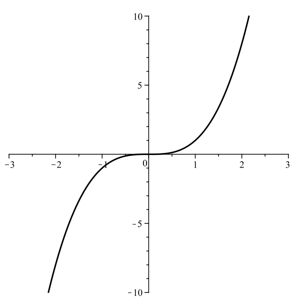
Square root function

$$f(x) = \sqrt{x}$$



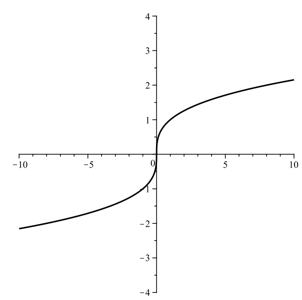
Cubing function

$$f(x) = x^3$$



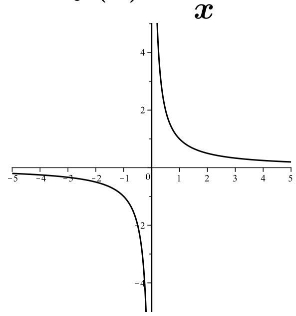
Cube root function

$$f(x) = \sqrt[3]{x}$$



Reciprocal function

$$f(x) = \frac{1}{x}$$



Transformations

Basically, when we talk about graph transformations, we are talking about *moving a graph* in the plane. We might move the graph up, down, left, or right; we might reflect the graph over the x-axis or over the y-axis; we may stretch or shrink the graph; or, we may do various combinations of these transformations.

We should be able to recognize how an equation or graph changes under a transformation, and we should be able to describe these transformations with words.

Example 5. Suppose $f(x) = x^2$. Describe $g(x) = x^2 + 3$ and $h(x) = (x + 2)^2$ relative to f, and graph all three functions on the same axes.

Since

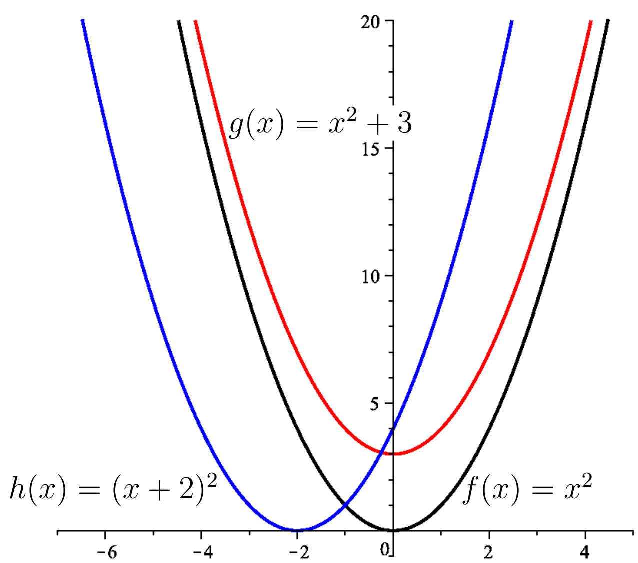
$$g(x) = x^2 + 3 = f(x) + 3,$$

the graph of g can be obtained by *vertically shifting* the graph of f **up** 3 units. The equation for g says that we should *add* 3 to all of the y-coordinates of f in order to get the graph of g.

Since

$$h(x) = (x + 2)^2 = f(x + 2),$$

the graph of h can be obtained by *horizontally shifting* the graph of f **left** 2 units. The equation for h says that we should *subtract* 2 from all of the x-coordinates of f in order to get the graph of h.



Exercise 1. Suppose $f(x) = \sqrt{x}$. Describe each of the following functions relative to f, and then graph the functions.

(i) $a(x) = -\sqrt{x}$

(ii) $b(x) = \sqrt{-x}$

(iii) $c(x) = 3\sqrt{x}$

(iv) $d(x) = \frac{1}{5}\sqrt{x}$

(v) $e(x) = \sqrt{x-3}$

(vi) $g(x) = \sqrt{x} - 2$

(vii) $h(x) = -2\sqrt{x+1} + 3$