

Section 1.6: The Algebra of Functions

Key points:

- Find the sum, difference, product, and quotient of two functions, and determine the domains of the resulting functions.
- Construct and simplify the difference quotient for a function.
- Find the composition of two functions and the domain of the composition.
- Decompose a function as a composition of two functions.

The Sum, Difference, Product, and Quotient of Two Functions

Suppose that f, g are two functions and x is in the domain of both of them. Then

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0.$$

Example 1. Suppose $f(x) = x + 3$ and $g(x) = x^2 + 2x$. Find the following:

(a) $f(-2)$, (b) $g(-2)$, (c) $(f + g)(-2)$, (d) $(fg)(-2)$, and (e) $\left(\frac{f}{g}\right)(-2)$.

$$(a) f(-2) = -2 + 3 = 1$$

$$(b) g(-2) = (-2)^2 + 2(-2) = 4 - 4 = 0$$

$$(c) (f + g)(-2) = f(-2) + g(-2) = 1 + 0 = 1$$

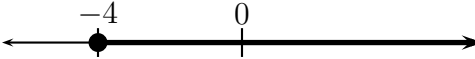
$$(d) (fg)(-2) = f(-2) \cdot g(-2) = 1 \cdot 0 = 0$$

$$(e) \left(\frac{f}{g}\right)(-2) = \frac{f(-2)}{g(-2)} = \frac{1}{0}, \text{ which is undefined, does not exist}$$

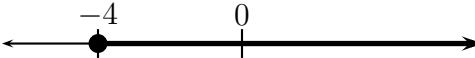
How to Determine the Domain of $f + g$, $f - g$, fg , and $\frac{f}{g}$

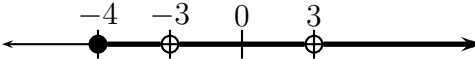
1. Find the domain of f .
2. Find the domain of g .
3. To find the domain of $f + g$, $f - g$, or fg , find where the domain of f and domain of g intersect, overlap, or have common values.
4. The domain of $\frac{f}{g}$ is the same as above, except any values of x which makes $g(x) = 0$ must be excluded (we do not allow division by zero!).

Example 2. Suppose $f(x) = \sqrt{x+4}$ and $g(x) = x^2 - 9$. Find the domains of $f + g$, $f - g$, fg , and $\frac{f}{g}$.

domain of f : $x + 4 \geq 0$ 
 $x \geq -4$

domain of g : $(-\infty, \infty)$ 

domain of $f + g$,
 $f - g$, fg : what do the
domain of f and
the domain of g
have in common? 

domain of $\frac{f}{g}$: same as above,
but must exclude
any values of x
which make
 $g(x) = 0$. 

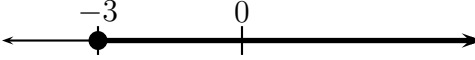
So, the domains of $f + g$, $f - g$, and fg are the same:

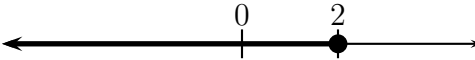
$$\{x \mid x \geq -4\} \quad \text{or} \quad [-4, \infty)$$

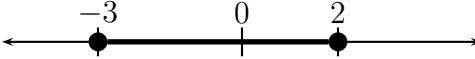
and since $g(x) = 0$ when $x = \pm 3$, the domain of $\frac{f}{g}$ is

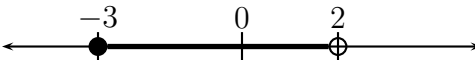
$$\{x \mid x \geq -4, x \neq \pm 3\} \quad \text{or} \quad [-4, -3) \cup (-3, 3) \cup (3, \infty).$$

Example 3. Suppose $f(x) = \sqrt{x+3}$ and $g(x) = \sqrt{2-x}$. Find the domain of $f+g$, $f-g$, fg , and $\frac{f}{g}$.

domain of f : $x+3 \geq 0$ 
 $x \geq -3$

domain of g : $2-x \geq 0$ 
 $2 \geq x$
 $x \leq 2$

domain of $f+g$,
 $f-g$, fg :  what do the domain of f and the domain of g have in common?

domain of $\frac{f}{g}$:  same as above, but must exclude any values of x which make $g(x) = 0$.

The domains of $f+g$, $f-g$, and fg are the same:

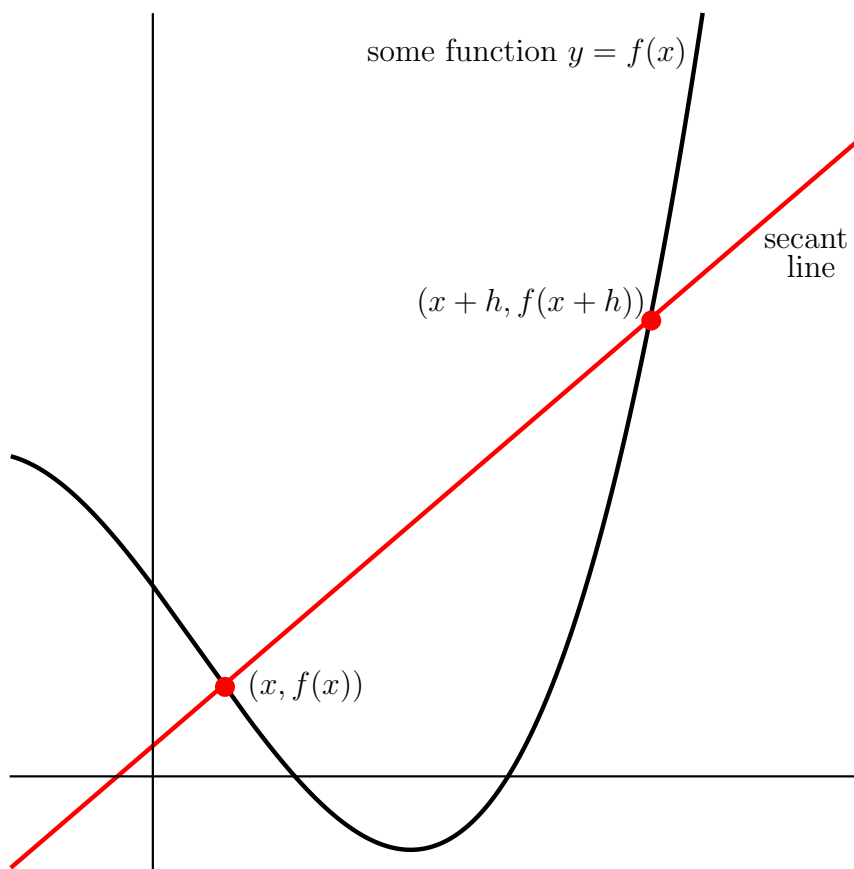
$$\{x \mid -3 \leq x \leq 2\} \quad \text{or} \quad [-3, 2]$$

and since $g(x) = 0$ when $x = 2$, the domain of $\frac{f}{g}$ is

$$\{x \mid -3 \leq x < 2\} \quad \text{or} \quad [-3, 2).$$

The Difference Quotient

Computation of the **difference quotient** is very important for students in algebra, and that skill will be used extensively in *calculus*. Consider the following figure:



The difference quotient is defined as

$$\begin{aligned} \text{difference quotient} &= \text{slope of the secant line} \\ &= \text{the average rate of change of } f \\ &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{f(x + h) - f(x)}{(x + h) - x} \\ &= \frac{f(x + h) - f(x)}{h}, \text{ where } h \neq 0. \end{aligned}$$

Example 4. Find the difference quotient for $f(x) = -3x + 2$.

$$\begin{array}{rcl}
 \text{Find } f(x+h): & f(x+h) = & -3(x+h) + 2 \\
 & & = -3x - 3h + 2 \\
 \text{Subtract } f(x): & -f(x) = & +3x \quad -2 \\
 \hline
 & f(x+h) - f(x) = & -3h
 \end{array}$$

Notice that the $-3x$ and $+3x$ cancel, as well as the $+2$ and -2 . Then the difference quotient is

$$\text{difference quotient} = \frac{f(x+h) - f(x)}{h} = \frac{-3h}{h} = -3, \text{ since } h \neq 0.$$

We are permitted to cancel the h 's **only** because we know that $h \neq 0$!

Example 5. Construct and simplify the difference quotient for

$$g(x) = -2x^2 + 3x + 5.$$

$$\begin{array}{rcl}
 \text{Find } g(x+h): & g(x+h) = & -2(x+h)^2 + 3(x+h) + 5 \\
 & & = -2(x+h)(x+h) + 3x + 3h + 5 \\
 & & = -2(x^2 + xh + hx + h^2) + 3x + 3h + 5 \\
 & & = -2(x^2 + 2xh + h^2) + 3x + 3h + 5 \\
 & & = -2x^2 - 4xh - 2h^2 + 3x + 3h + 5 \\
 \text{Subtract } g(x): & -g(x) = & +2x^2 \quad -3x \quad -5 \\
 \hline
 & g(x+h) - g(x) = & -4xh - 2h^2 + 3h,
 \end{array}$$

after canceling like terms. Then the difference quotient is

$$\frac{g(x+h) - g(x)}{h} = \frac{-4xh - 2h^2 + 3h}{h} = \frac{h(-4x - 2h + 3)}{h} = -4x - 2h + 3,$$

where $h \neq 0$. Again, we are permitted to cancel the h 's **only** because we know that $h \neq 0$!

Composition of Functions

If we think of a function f as a machine, where we give the function the input value x and it returns the output value $f(x)$, then composition of functions is like an assembly line.

Every morning, most of you experience compositions of functions in your daily lives, for example, while you are getting dressed. First, you put on your socks, and then you put on your shoes. These are two distinct processes, one following the other.

Composition of functions is no different. The **composition of f with g** is denoted

$$(f \circ g)(x) = f(g(x)),$$

where x is in the domain of g and $g(x)$ is in the domain of f .

The little circle “ \circ ” is read as “composed with”, so that the above expression is read as “ f composed with g at x equals f of g of x .” The way that this function composition “works” is that g is given input value x , to produce $g(x)$, and then f is given input value $g(x)$ to produce $f(g(x))$.

In the composition $(f \circ g)(x)$, some people like to think of g as the “inner” function because it is applied first and f as the “outer” function since it is applied last.

Example 6. Suppose $f(x) = 3x + 1$ and $g(x) = x^2 - 5$. (a) Write out, in words, what f and g do to their inputs, (b) compute $(f \circ g)(x)$, (c) compute $(g \circ f)(x)$, (d) evaluate $(f \circ g)(2)$, and (e) evaluate $(g \circ f)(-1)$.

(a) f multiplies the input by 3, then adds one

and

g squares the input, then subtracts 5

(b)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(x^2 - 5) = 3(x^2 - 5) + 1 \\ &= 3x^2 - 15 + 1 \\ &= 3x^2 - 14\end{aligned}$$

All that we did was apply the definition of function composition, replace $g(x)$ with its equation $x^2 - 5$, and then apply the function f .

$$\begin{aligned}
\text{(c)} \quad (g \circ f)(x) &= g(f(x)) = g(3x + 1) = (3x + 1)^2 - 5 \\
&= (3x + 1)(3x + 1) - 5 \\
&= 9x^2 + 3x + 3x + 1 - 5 \\
&= 9x^2 + 6x - 4
\end{aligned}$$

All that we did was apply the definition of function composition, replace $f(x)$ with its equation $3x + 1$, and then apply the function g .

$$\text{(d)} \quad (f \circ g)(2) = 3(2)^2 - 14 = 3 \cdot 4 - 14 = -2$$

$$\text{(e)} \quad (g \circ f)(-1) = 9(-1)^2 + 6(-1) - 4 = 9 \cdot 1 - 6 - 4 = -1$$

Example 7. Suppose $h(x) = \sqrt[3]{2x^2 + 5}$. (a) Write out, in words, what h does to its input and (b) find two functions f and g such that $h(x) = (f \circ g)(x)$.

(a) h squares its input, multiplies the result by 2, adds 5, then takes the cube root

(b) The last thing that h does is take the cube root, so define the “outer function” f by

$$f(x) = \sqrt[3]{x}$$

and the “inner function” g by the parts of h that are “left over”:

$$g(x) = 2x^2 + 5.$$

It is easy to verify that

$$(f \circ g)(x) = f(g(x)) = f(2x^2 + 5) = \sqrt[3]{2x^2 + 5} = h(x).$$

Example 8. Page 145, #104.