RANDOMLY STAR-DECOMPOSABLE GRAPHS (*)

Christian Barrientos, Aldo Bernasconi,
Eric Jeltsch, Claudio Troncoso
Universidad Técnica Federico Santa María, Chile

Sergio Ruiz
Universidad Católica de Valparaíso, Chile.

ABSTRACT

Given a graph H, an H-decomposable graph G is said to be randomly H-decomposable if every family of edge-disjoint subgraphs of G, each subgraph isomorphic to H, can be extended to an H-decomposition of G. A characterization of randomly H-decomposable graphs is presented, where H is the star K(1,n).

A decomposition F of a nonempty graph G is a collection of subgraphs $H_1, H_2, \ldots, H_k$ such that their edge sets $E(H_1)$, $E(H_2), \ldots, E(H_k)$ form a partition of E(G). Such a decomposition is called an <u>H-decomposition</u> if each of the graphs $H_1, H_2, \ldots, H_k$ is isomorphic to a given graph H. A graph having an H-decomposition is said to be <u>randomly H-decomposable</u> if any collection of edge disjoint subgraphs of G, each isomorphic to H, can be extended to an H-decomposition of G.

In [1], Ruiz introduced the concept of randomly decomposable graphs along with the characterization of those graphs which are H-decomposable for the case where H has two edges.

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If a graph G is randomly H-decomposable then an H-decomposition can easily be found. In each step, simply choose a subgraph isomorphic to H, none of whose edges was chosen previously.

In this paper, the stars K(1,n) will be denoted by S_n. For a given positive integer n, we will determine all those graphs which are randomly S_n-decomposable.

THEOREM

A nontrivial graph G is randomly S_n-decomposable ($n \geq 2$) if and only if each component of G is either

i) isomorphic to K(n,n) or

ii) isomorphic to a bipartite graph with bipartition $\{V_0, V_1\}$ such that the vertices of V_0 have degrees which are multiple of n and the vertices of V_1 have degrees less than n.

PROOF

Since S_n is connected, a graph is randomly S_n-decomposable if and only if each of its components is randomly S_n-decomposable. Thus, we may assume that G is connected.

Sufficiency:

If $G \cong K(n,n)$ then by deleting from G the edges of any copy of S_n, it is obtained a graph satisfying condition ii). Thus, without loss of generality we suppose that G is connected and satisfies ii). All subgraphs of G isomorphic to S_n must have their centers at V_0 and an inductive argument makes apparent that G is randomly S_n-decomposable.

Necessity

Suppose that G is a randomly S_n-decomposable connected graph. First, note that G is bipartite. To see this, consider a S_n-decomposition F of G. Define as V_0, the set of vertices of G which are centers of stars in F and define $V_1 = V(G)-V_0$.

We will show that V_0 and V_1 are both independent sets. If u and v were two adjacent vertices of V_0, then the edge uv belongs to a star A in F, centered at u, say. Since $v \in V_0$, there must be another star B centered at v. Now, consider the star $C \notin F$, centered at v, formed by selecting the edge uv and n-1 of the remaining edges of B. But the set of stars

$$F' = (F - \{A,B\}) \cup \{C\}$$

cannot be extended to a S_n-decomposition, since the graph obtained from G by removing the edges of all members in F' is isomorphic to $S_{n-1} \cup K_2$. Therefore, V_0 is an independent set. Now, suppose that x and y are two adjacent vertices of V_1. Since the edge xy must be an edge of some star in F, we conclude that either x or y is a center of a star in F, which is a contradiction, therefore V_1 is also an independent set. Thus, G is bipartite and the centers of the stars S_n in a given S_n-decomposition of G, belong to the same partite set.

So far, we have that G is a bipartite graph. Assume further that G is not isomorphic to K(n,n). Consider a particular S_n-decomposition of G where the centers form the set V_0. If all vertices in $V_1 = V(G) - V_0$ have degree less than n, then we are done. The proof will be complete when we find a contradiction under the assumption that there is a vertex v in V_1 with degree $\geq n$. By choosing a star S_n centered at v, it is possible to extend this star to a S_n-decomposition of G, so that, actually all vertices in V_1 become centers of stars. This implies that all vertices in V_1 also have degrees which are multiple of n.

Now, since G is not isomorphic to K(n,n), we can find a star S_n centered at a vertex in V_0 and another star centered at a vertex in V_1 such that these two stars do not share any common edge. Certainly, these two stars cannot be extended to a S_n-decomposition of G. A contradiction.

REFERENCE

[1] S. Ruiz, Randomly decomposable graphs, Discrete Math. 57 (1985) 123-128.