Maximum Demand Graphs for Eternal Security

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Abstract

Informally, a set of guards positioned on the vertices of a graph G is called eternally secure if it is able to respond to any sequence of vertex attacks by moving a single guard along a single edge after each individual attack. The smallest number of guards required is the eternal security number of G, denoted $es(G)$, and it is known to be no more than $\theta_v(G)$, the vertex clique cover number of G. We investigate conditions under which $es(G) = \theta_v(G)$.

Keywords: domination, eternal security

1 Introduction

Burger, Cockayne, Gründlingh, Mynhardt, van Vuuren, and Winterbach [1, 2], introduced a dynamic form of domination which has been designated *eternal security* by Goddard, Hedetniemi, and Hedetniemi [4]. The concept calls for a fixed number of guards which are positioned on the vertices of graph $G = (V, E)$, at most one to a vertex. A guard on vertex w can respond to an attack on a vertex v by moving along an edge from w to v (assuming v does not already have a guard). Informally, if such a response can be made no matter what vertex is attacked, and if the changing position of the guards can continue to respond forever, we say that the guards form an eternally secure set.

In order to formalize this concept, Goddard, et al [4] gave the following definition.

Definition 1 *Let $S_0 \subseteq V$. Then S_0 is an* eternally secure set *if for every positive integer k and any sequence $v_1, v_2, \ldots, v_k$ of vertices of G, there is a sequence of sets $S_1, S_2, \ldots, S_k$ and a sequence of vertices (guards) $u_1, u_2, \ldots, u_k$ such that $v_i \notin S_{i-1}$, $u_i \in S_{i-1}$, $u_iv_i \in E$, and $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is a dominating set. The* eternal security number *$es(G)$ of G is the minimum cardinality of an eternally secure set and an* es-set *of G is an eternally secure set S such that $|S| = es(G)$.*

References [2] and [4] employ the notation $\gamma_\infty(G)$ and $\sigma_1(G)$, respectively, for $es(G)$.

Definition 1 allows a possible misinterpretation. To obtain the desired form of eternal security, one must not assume that the attack sequence is fixed, since if it is

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then the required number of guards can sometimes be reduced. As an example consider C_5. It is shown in [2] that $es(C_5) = 3$. Suppose, however, that the attack sequence is fixed. Let the vertices be in order $u_1, u_2, \ldots, u_5$ and $v_1, v_2, \ldots, v_k$ be a fixed sequence of vertex attacks. Assign guards to u_1 and u_3. If $v_1 = u_4$ or $v_1 = u_5$, respond with the guard at u_3 or u_1, respectively, which creates a defense isomorphic to the original. If $v_1 = u_2$, let i be the smallest index such that $v_i \notin \{u_1, u_2, u_3\}$. If $v_i = u_4$, respond to the attack on u_2 with the guard at u_1 and use that guard to respond to all attacks on u_1 and u_2 until the i^{th} attack on u_4 which is responded to by the guard on u_3. After the i^{th} response, the defense is again isomorphic to the original. A similar strategy is employed if $v_i = u_5$. Thus only two guards are required.

Motivated by an approach mentioned in [4], we avoid the above difficulty by employing an algorithmic definition. The input to the algorithm is a graph $G = (V, E)$. For $S \in P(V(G))$, where $P(V(G))$ is the power set of $V(G)$, and $v \in V(G) - S$ we define a *response by* S (or simply a *response* if S is understood) to an attack on v to be a vertex $w \in S$ such that w is adjacent to v. We say response w to an attack on v *produces* the set $S' = (S - w) \cup \{v\}$. The following algorithm defines a function μ_G on a subset of $P(V(G))$ which will be important in later proofs.

ALGORITHM ES

$\mathcal{E} := P(V(G))$
$\mathcal{N} := \emptyset$
$\mathcal{T} := \{S \in \mathcal{E}: S$ not a dominating set$\}$
$\mu_G(S) := 0$ for all $S \in \mathcal{T}$
$k := 0$
while $\mathcal{T} \neq \emptyset$
    $k := k + 1$
    $\mathcal{N} := \mathcal{N} \cup \mathcal{T}$
    $\mathcal{E} := \mathcal{E} - \mathcal{T}$
    $\mathcal{T} := \{S \in \mathcal{E}$: there is an attack on some vertex such that all responses produce a set in $\mathcal{N}\}$
    $\mu_G(S) := k$ for all $S \in \mathcal{T}$

Note that if $\mu_G(S) > 0$, there exists a vertex v such that $\mu_G(S') < \mu_G(S)$ for every S' produced by a response to an attack on v. The algorithm must terminate since $P(V(G))$ is finite. Furthermore, $\mathcal{E}$ is nonempty upon termination since V itself will never be placed in $\mathcal{N}$. We now can give the revised definition of eternally secure sets.

Definition 2 *Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is* eternally secure *if it remains in $\mathcal{E}$ upon termination of ALGORITHM ES when applied to G. The value $es(G)$ is the smallest cardinality of a set in $\mathcal{E}$.*

It is possible to view this attack/response process as a two person game in which one player selects a vertex to attack and the other decides on a response. Then, for a starting defense S of guards which is not eternally secure, $\mu_G(S)$ is a measure of the number of steps in the game, assuming optimum play by both participants.

The value of $\mu_G(S)$ found by ALGORITHM ES can also be determined by the following recursive definition.

Definition 3 *Let $G = (V, E)$ be a graph. For a non eternally secure set $S \subseteq V$*

$$\mu_G(S) = \begin{cases} 0 & \text{if } S \text{ is nondominating} \\ k & \text{if there is an attack on some vertex such that} \\ & \quad \text{for any response } S',\ \mu_G(S') \leq k-1, \text{and} \\ & \quad \text{for some response } S',\ \mu_G(S') = k-1 \end{cases}$$

In subsequent discussions we may speak informally of responses to attacks in terms of the movement of a guard. In most such cases the original and final sets of guards form eternally secure sets.

Let $\beta_v(G)$ be the vertex independence number of graph G and $\theta_v(G)$ be its vertex clique cover number. Burger et al show the following basic inequality.

Theorem 4 [2] *For graph G, $\beta_v(G) \leq es(G) \leq \theta_v(G)$.*

Many important classes of graphs have $es(G) = \theta_v(G)$ and it was originally conjectured that this is true for all graphs. However, Goddard et al [4] have produced an infinite collection of counterexamples. Nevertheless, it is interesting to find families where this equality holds. We refer to members of such families as *maximum-demand graphs.*

References [2] and [4] present many examples of maximum-demand graphs. These include paths, cycles, bipartite graphs, complete multipartite graphs, perfect graphs, and graphs for which the vertex clique cover number is at most three. Other examples include the Cartesian products $K_m \times K_n$, $P_m \times P_n$, $C_4 \times C_5$, and $C_5 \times C_5$. Burger et al [2] prove $7mn/23 \leq es(C_m \times C_n) \leq \theta_v(C_m \times C_n)$ in general. In Section 2 we show $C_m \times C_n$ is a maximum-demand graph for all values of m and n. Section 3 discusses the situation for the coalescence of two graphs, Section 4 presents results pertaining to a minimum non maximum-demand graph, and Section 5 shows that graphs with no subgraph homeomorphic to K_4 are maximum-demand graphs. We conclude with some open questions.

2 Eternal Security Number of $C_m \times C_n$ and $P_m \times C_n$

If m and n are both even, $C_m \times C_n$ is bipartite and thus is a maximum-demand graph. It is straightforward to demonstrate $C_m \times C_n$ is a maximum-demand graph for $n \leq 3$ and $m \geq n$. Thus we may assume $n \geq 4$ and $m \geq 5$ is odd. The following theorem solves all the remaining cases.

Theorem 5 *Let n and m be integers where $n \geq 4$ and $m \geq 5$ is odd. Then $C_m \times C_n$ is a maximum-demand graph.*

Proof: It is easy to see that $\theta_v(C_m \times C_n) = \lceil \frac{mn}{2} \rceil$. We claim that $es(C_m \times C_n)$ has the same value. Without loss of generality assume that either n is even or n is odd and $n \geq m$. Embed the vertices of $C_m \times C_n$ on the m by n lattice $\{(i,j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ such that the vertices in any row or column form a cycle of $C_m \times C_n$. Designate the vertex at (i,j) by $v_{i,j}$. For $0 \leq t \leq m-1$, let $d_t = \{v_{t,0}, v_{t+1,1}, v_{t,2}, v_{t+1,3}, v_{t,4}, \ldots, v_{t+1,n-1}\}$ if n is even and $d_t = \{v_{t,0}, v_{t+1,1}, \ldots, v_{t+m-1,m-1}, v_{t+m-2,m}, v_{t+m}$ if n is odd. In both cases the addition in the first subscript is modulo m. Figure 1 gives two examples, with the vertices of d_0 marked with circles and those of d_2 with squares. Note that $d_t \cup d_{t+2}$, with the subscript modulo m, is an independent set and the closed neighborhood of d_1 is $d_0 \cup d_1 \cup d_2$. First, attack $d_0 \cup d_2$, forcing a response containing all of these $2n$ vertices. Next, attack the vertices of d_1. All responses must come from $d_0 \cup d_1 \cup d_2$. Without loss of generality, $\lceil \frac{n}{2} \rceil$ of the guards in d_0 do not respond to these attacks. Thus, there are at least $n + \lceil \frac{n}{2} \rceil$ guards on $d_0 \cup d_1$ in the resulting response. Attack the vertices in $d_3, d_5, ...d_{m-2}$. These vertices form an independent set and are disjoint from the closed neighborhood of $d_0 \cup d_1$; hence, all of these $\frac{n(m-3)}{2}$ vertices must be in the resulting response. All together there are at least $n + \lceil \frac{n}{2} \rceil + \frac{n(m-3)}{2} = \lceil \frac{mn}{2} \rceil$ vertices in the final response. □

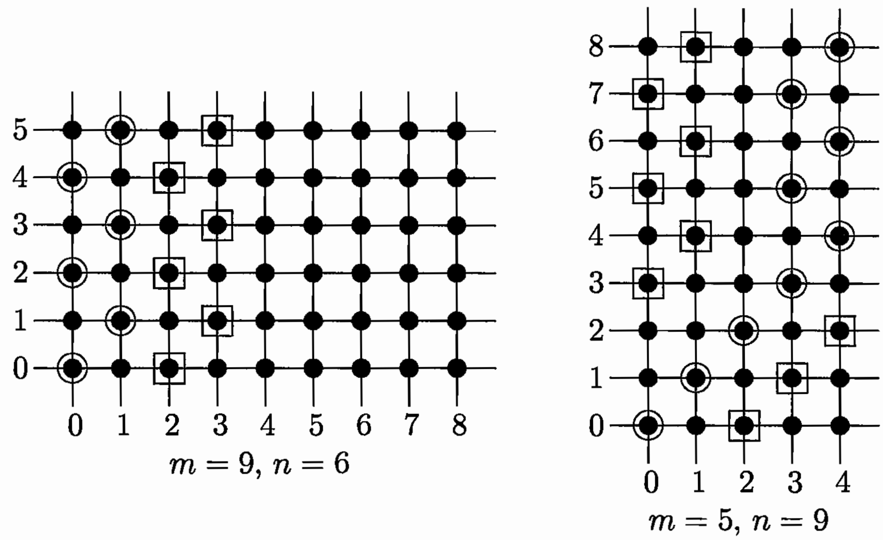

Figure 1: Examples showing sets d_0 and d_2

The following general observation is useful in determining $es(P_m \times C_n)$.

Observation 6 *If H is a spanning subgraph of graph G, then $es(H) \geq es(G)$.*

Using this observation and the fact than $\theta_v(P_m \times C_n) = \theta_v(P_m \times P_n) = \theta_v(C_m \times C_n)$, we have the following result.

Theorem 7 *The graphs $P_m \times C_n$ and $P_m \times P_n$ are maximum-demand graphs.*

Of course, the fact that $P_m \times P_n$ is a maximum-demand graph was known previously. [4]

3 On the Coalescence of Two Graphs

Let H and K be graphs with designated vertices v_H and v_K, respectively. The *coalescence* G of H and K, written $G = H \circ K$, is the graph obtained by identifying v_H and v_K. We designate the common vertex in G by v, and, for convenience, also employ v for v_H and v_K when discussing either H or K alone. Furthermore we call the copies of H and K in G by the same symbols H and K, respectively.

Lemma 8 *Let v be a vertex in a graph G. Then $es(G - v) \geq es(G) - 1$.*

Proof: Any eternally secure set of $G - v$ along with v is an eternally secure set of G. □

Lemma 9 *Let $G = H \circ K$. Then $\theta_v(H) + \theta_v(K) - 1 \leq \theta_v(G) \leq \theta_v(H) + \theta_v(K)$ and $es(H) + es(K) - 1 \leq es(G) \leq es(H) + es(K)$.*

Proof: The upper bounds are immediate since the unions of es-sets or vertex clique covers for H and K forms an eternal security set or vertex clique cover, respectively, of G. The lower bound for $\theta_v(G)$ follows since it can be smaller than $\theta_v(H) + \theta_v(K)$ only if v is in a singleton clique of a minimum vertex clique cover of either H or K, and then the total number of cliques required is reduced only by one from $\theta_v(H) + \theta_v(K)$.

Suppose $es(G) \leq es(H) + es(K) - 2$ and let S be an es-set of G. Note that only vertices of $S \cap V(K)$ can respond to attacks on $K - v$, so, $|S \cap V(K)| \geq es(K - v)$. Thus, by Lemma 8, $|S \cap V(K)| \geq es(K) - 1$ and similarly $|S \cap V(H)| \geq es(H) - 1$. The upper bound on $es(G)$ implies at least one of the inequalities must be an equality. Without loss of generality, we assume $|S \cap V(H)| = es(H) - 1$ which implies $S \cap V(H)$ is not an eternally secure set for H. Among all such sets we choose S so that $\mu_H(S \cap V(H))$ is a minimum. Since $S \cap V(H)$ is not an eternally secure set for H there exists a vertex $x \in V(H) - S$ such that either (i) $S \cap V(H)$ does not dominate x or (ii) for every S' produced by a response to an attack on x, $\mu_H(S') < \mu_H(S \cap V(H))$. Thus, no vertex of $S \cap V(H)$ can respond to an attack on x without contradicting the definition of S. If $x \neq v$, then all responses from S to attacks on x must come from $S \cap V(H)$ and we contradict S being an es-set for G. Therefore, $v = x \notin S$ and $S \cap V(H) = S \cap V(H - v)$. If now we restrict attacks to K, then the vertices of $S \cap V(H)$ cannot respond. Therefore, $S - (S \cap V(H))$ must be an eternally secure set for K. Again we have a contradiction since $|S - (S \cap V(H))| \leq (es(H) + es(K) - 2) - (es(H) - 1) = es(K) - 1$. □

We show next that if H and K are maximum-demand graphs, then there is a condition under which $H \circ K$ is a maximum-demand graph.

Proposition 10 *Let $G = H \circ K$ where H, K, $H - v$, and $K - v$ are all maximum-demand graphs. Then G is a maximum-demand graph.*

Proof: By Lemma 9 we may restrict our attention to the case $es(G) = es(H) + es(K) - 1$ and $\theta_v(G) = \theta_v(H) + \theta_v(K)$. Observe that no minimum vertex clique cover of either H or K contains v in a singleton set. Let S be an es-set of G.

Case 1. Suppose $v \notin S$. The condition $es(G) = es(H) + es(K) - 1$ implies either $|S \cap V(H)| \leq es(H) - 1$ or $|S \cap V(K)| \leq es(K) - 1$. Assume without loss of generality that $|S \cap V(H)| \leq es(H) - 1$. By construction, no vertex of $K - v$ can respond to an attack on $H - v$; so, $v \notin S$ implies that $S \cap V(H) = S \cap V(H - v)$ is an eternally secure set for $H - v$. Thus, $es(H - v) \leq es(H) - 1$. Applying our assumptions, we obtain $\theta_v(H - v) = es(H - v) \leq es(H) - 1 = \theta_v(H) - 1$. Therefore, we can add $\{v\}$ to a minimum clique cover of $H - v$ to obtain a minimum clique cover of H, contrary to the above observation.

Case 2. Next suppose $v \in S$. Here the condition $es(G) = es(H) + es(K) - 1$ implies either $|S \cap V(H)| \leq es(H)$ or $|S \cap V(K)| \leq es(K)$. Assume without loss of generality that $|S \cap V(H)| \leq es(H)$. If v is required to respond at some point to an attack on $H - v$, we are reduced to Case 1. Otherwise, $S \cap V(H - v)$ is an eternally secure set for $H - v$ and $es(H-v) \leq |S \cap V(H-v)| \leq es(H) - 1$. This leads to the same contradiction as in the first case. □

Proposition 10 is not valid if the condition that $H - v$ and $K - v$ are maximum-demand graphs is removed. One of the examples due to Goddard, Hedetniemi, and Hedetniemi [4] shows that $3 = es(G) < \theta_v(G) = 4$ if G is the complement of the Grötzsch graph shown in Figure 2. We employ this fact in showing that the proposition does not extend.

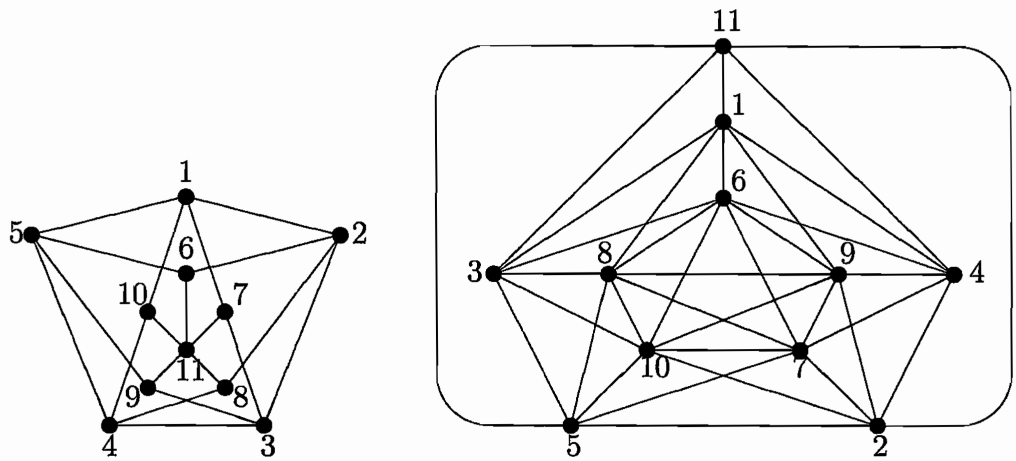

Figure 2: The Grötzsch graph and its complement

Create a graph H by adding a pendant vertex v adjacent to vertex 4 in the complement of the Grötzsch graph. Because v, 1, and 5 are independent, there must exist an es-set S of H that contains them. Suppose that $S = \{v, 1, 5\}$. Attack vertex 3. Vertex v cannot respond since it is not adjacent to vertex 3. If vertex 1 responds, vertex 9 is not dominated. If vertex 5 responds, vertex 2 is not protected. Thus $es(H) \geq 4$. It is easy to see that $\theta_v(H) = 4$, so $es(H) = 4$. Let $K = H$ and form G as the coalescence of H and K with the vertex v's identified. Then three vertices form an eternal security set of $H - v$ and four of K. It follows that $es(G) = 7$ but $\theta_v(G) = 8$ so Proposition 10 does not extend. Notice that if F is the graph formed by coalescing n copies of H at the vertex v, $es(F) = 3n + 1$ and $\theta_v(F) = 4n$, so the difference in the two values can be arbitrarily large. This arbitrary difference also can be shown by the examples in [4].

4 Adding Paths

This section is concerned with when it is possible to construct maximum-demand graphs by adding paths to a known maximum-demand graph. The following four lemmas provide useful tools.

Lemma 11 *Suppose H is an induced subgraph of graph G. If S is an eternally secure set of G such that $|S \cap V(H)|$ is maximized, then $S \cap V(H)$ is an eternally secure set of H.*

Proof: Suppose not. Choose S to be an eternally secure set of G such that $|S \cap V(H)|$ is maximized, $S \cap V(H)$ is not an eternally secure set of H, and $\mu_H(S \cap V(H))$ is minimized. Since $S \cap V(H)$ is not an eternally secure set of H, there exists a vertex $x \in H$ such that either (i) $S \cap V(H)$ does not dominate x or (ii) if $w \in S \cap V(H)$ is a response to an attack on x, then $\mu_H(((S \cap V(H)) \cup x) - \{w\}) < \mu_H(S \cap V(H))$. The choice of S precludes (ii); hence, (i) must hold and no vertex of $S \cap V(H)$ can respond to an attack on x. Therefore, the response from S to an attack on x must come from $G - H$ which contradicts the maximality of $|S \cap V(H)|$. □

Lemma 12 *Suppose graph G is obtained from graph H by adding a path P of length at least 2 between distinct vertices v and w. Suppose v and w are contained in every es-set S of G such that $|S \cap V(H)|$ is maximized. Either $|S \cap V(H)| \geq es(H) + 1$ or $es(H - \{v, w\}) = es(H) - 2$. Furthermore, if v and w are adjacent, $|S \cap V(H)| \geq es(H) + 1$.*

Proof: Suppose v and w are contained in every es-set S of G such that $|S \cap V(H)|$ is maximized and let S be such a set. By Lemma 11, $|S \cap V(H)| \geq es(H)$. Suppose $|S \cap V(H)| = es(H)$. If we restrict our attacks to $H - \{v, w\}$, then v and w cannot respond without violating the assumption that they are in every es-set with maximum intersection with H. This implies that $S \cap V(H - \{v, w\})$ is an eternally secure set for $H - \{v, w\}$. Thus, $es(H - \{v, w\}) \leq |S \cap V(H - \{v, w\})| = |S \cap V(H)| - 2 = es(H) - 2$. Since adding $\{v, w\}$ to any eternally secure set of $H - \{v, w\}$ produces an eternally secure set for H, we have $es(H - \{v, w\}) + 2 \geq es(H)$; hence, $es(H - \{v, w\}) = es(H) - 2$. If v and w are adjacent, then $S \cap V(H - v)$ is an eternally secure set of H which implies $|S \cap V(H)| \geq es(H) + 1$. □

The proof of Lemma 12 actually suffices for any cutset $\{v, w\}$ but we do not require that result here.

Lemma 13 *Let $n \geq 3$ and suppose G is obtained from graph H by adding a path $P = \langle v_1, v_2, ..., v_n \rangle$ between distinct vertices v_1 and v_n of H. Let S be an es-set for G. If $|S \cap V(H)| \geq es(H) + 1$ then $es(G) \geq es(H) + \lceil \frac{n}{2} \rceil - 1$.*

Proof: Attacks on the $\lceil \frac{n}{2} \rceil - 2$ independent vertices $v_3, v_5, ..., v_{2\lceil \frac{n}{2} \rceil - 3}$ cannot be responded to by vertices in H. Thus $|S \cap V(G - H)| \geq \lceil \frac{n}{2} \rceil - 2$ and $es(G) = |S| \geq es(H) + 1 + \lceil \frac{n}{2} \rceil - 2 = es(H) + \lceil \frac{n}{2} \rceil - 1$. □

Lemma 14 *Suppose $n \geq 4$ is an even integer and G is obtained from graph H by adding a path $P = \langle v_1, v_2, ..., v_n \rangle$ between distinct vertices v_1 and v_n of H. If there exists an es-set S for G such that $S \cap V(H)$ is maximized and $|S \cap \{v_1, v_n\}| \leq 1$, then $es(G) \geq es(H) + \frac{n}{2} - 1$.*

Proof: Without loss of generality suppose v_1 is not in S. By Lemma 11, $S \cap V(H)$ is an eternally secure set for H, so $|S \cap V(H)| \geq es(H)$. Attacks on the independent vertices $v_2, v_4, \ldots, v_{n-2}$ must be responded to by vertices in $S - V(H)$ which implies $|S - V(H)| \geq \frac{n}{2} - 1$. Thus, $es(G) = |S| = |S \cap V(H)| + |S - V(H)| \geq es(H) + \frac{n}{2} - 1$. □

Theorem 15 *Let $n \geq 4$ and suppose G is obtained from graph H by adding a path $P = \langle v_1, v_2, ..., v_n \rangle$ between distinct vertices v_1 and v_n of H. If H, $H - v_1$, $H - v_n$, and $H - \{v_1, v_n\}$ are maximum-demand graphs, then G is a maximum-demand graph.*

Proof: Adding $\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{2\lceil\frac{n}{2}\rceil-2}, v_{2\lceil\frac{n}{2}\rceil-1}\}$ to a clique cover of H provides a clique cover of G, so $\theta_v(G) \leq \theta_v(H) + \lceil\frac{n}{2}\rceil - 1$. If $es(G) \geq es(H) + \lceil\frac{n}{2}\rceil - 1$, the theorem follows from the assumption $es(H) = \theta_v(H)$ and the inequality $es(G) \leq \theta_v(G)$. Consequently, we assume that $es(G) \leq es(H) + \lceil\frac{n}{2}\rceil - 2$. Let S be an es-set for G such that $|S \cap V(H)|$ is maximized. By Lemma 11, $S \cap V(H)$ is an eternally secure set of H. By the contrapositive to Lemma 13, $|S \cap V(H)| \leq es(H)$; hence, $|S \cap V(H)| = es(H)$ and $|S - V(H)| \leq \lceil\frac{n}{2}\rceil - 2$. Also attacks on the $\lceil\frac{n}{2}\rceil - 2$ independent vertices $v_3, v_5, \ldots, v_{1+2(\lceil\frac{n}{2}\rceil-2)}$ can be responded to only by vertices in $S - V(H)$; hence, $|S - V(H)| = \lceil\frac{n}{2}\rceil - 2$ and $es(G) = es(H) + \lceil\frac{n}{2}\rceil - 2$.

If n is even, the contrapositive to Lemma 14 implies $|S \cap \{v_1, v_n\}| \geq 2$ and hence the hypothesis of Lemma 12 is satisfied. Thus, by Lemma 12, $es(H - \{v_1, v_n\}) = es(H) - 2$. Adding $\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2\lceil\frac{n}{2}\rceil-1}, v_{2\lceil\frac{n}{2}\rceil}\}$ to a clique cover of $H - \{v_1, v_n\}$ provides a clique cover of G; so, $\theta_v(G) \leq \theta_v(H - \{v_1, v_n\}) + \lceil\frac{n}{2}\rceil = es(H - \{v_1, v_n\}) + \lceil\frac{n}{2}\rceil = es(H) - 2 + \lceil\frac{n}{2}\rceil = es(G)$ and the theorem follows in this case.

Suppose n is odd. If every such S contains both v_1 and v_n, then the hypothesis of Lemma 12 holds and we may proceed as in the even case replacing $\{v_{2\lceil\frac{n}{2}\rceil-1}, v_{2\lceil\frac{n}{2}\rceil}\}$ with $\{v_{2\lceil\frac{n}{2}\rceil-1}\}$. Otherwise, we may assume without loss of generality that v_1 is not in S. Obtain S' from S by attacking the $\lceil\frac{n}{2}\rceil - 2$ independent vertices $v_2, v_4, \ldots, v_{n-3}$. Since $v_1 \notin S$ all of these attacks must be responded to by vertices in $S - V(H)$. Thus $|S' \cap V(H)| = |S \cap V(H)| = es(H)$ and $|S' - V(H)| = |S - V(H)| = \lceil\frac{n}{2}\rceil - 2$; hence $S' - V(H) = \{v_2, v_4, \ldots, v_{n-3}\}$. Thus, v_n must be in S' in order to dominate v_{n-1}, and furthermore it cannot respond to attacks on $H - v_n$ without leaving v_{n-1} unguarded. It follows that $S' \cap V(H - v_n)$ is an eternally secure set for $H - v_n$. Hence, $es(H - v_n) \leq es(H) - 1$. By assumption $\theta_v(H - v_n) = es(H - v_n)$. Also, adding the $\lceil\frac{n}{2}\rceil - 1$ cliques $\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{n-1}, v_n\}$ to a clique cover of $H - v_n$ produces a clique cover of G. Therefore, $\theta_v(G) \leq \theta_v(H - v_n) + \lceil\frac{n}{2}\rceil - 1 \leq es(H) - 1 + \lceil\frac{n}{2}\rceil - 1 = es(G)$ and the theorem follows. □

Lemma 16 *Let G be obtained from graph H by adding a path $P = \langle v_1, v_2, v_3 \rangle$ between adjacent vertices v_1 and v_3. If H and $H - \{v_1, v_3\}$ are maximum-demand graphs, then G is a maximum-demand graph.*

Proof: Clearly $\theta_v(G) \leq \theta_v(H) + 1$ and $es(H) \leq es(G) \leq es(H) + 1$. If $es(G) = es(H) + 1$, the result follows from the assumption that $es(H) = \theta_v(H)$. Thus suppose $es(G) = es(H)$. Since v_1 and v_3 are adjacent, the contra-positive to Lemma 12 implies there exists an es-set S of G such that $|S \cap V(H)|$ is maximized and either v_1 or v_3 is not in S. Without loss of generality we assume $v_1 \notin S$. Since $|S \cap V(H)| = es(H) = es(G) = |S|$, the vertex $v_2 \notin S$. This implies that $v_3 \in S$ or v_2 would be unguarded. Note that v_3 cannot respond to any attack on $H - P$ without leaving v_2 unguarded. Thus, $S \cap V(H - P)$ must be an eternally secure set for $H - P$ which implies $es(H - P) \leq es(H) - 1$. By assumption, $\theta_v(H - P) = es(H - P)$. Thus $\theta_v(H - P) \leq es(H) - 1 = \theta_v(H) - 1$. Adding the clique $\{v_1, v_3\}$ to any vertex clique cover of $H - P$ creates a vertex clique cover of H, so $\theta_v(H) \leq \theta_v(H - P) + 1$. We conclude that $\theta_v(H) = \theta_v(H - P) + 1$ and there exists a minimum clique cover C of H such that $\{v_1, v_3\} \in C$. Then, $(C - \{v_1, v_3\}) \cup \{v_1, v_2, v_3\}$ is a clique cover of G of size $\theta_v(H)$. Thus, $\theta_v(G) \leq \theta_v(H) = es(H) = es(G)$ and the result follows. □

It clearly is not true in general that $es(G) = \theta_v(G)$ if G is obtained from H by adding a path of length one between two vertices of H, even if every subgraph K of H satisfies $es(K) = \theta_v(K)$. Simply consider the complement of the Grötzsch graph minus an edge. The only remaining case to consider is adding a path of length two between two independent vertices. This case remains open.

5 Eternal Security Number of K_4 Minor Free Graphs

A graph is said to be K_4 minor free if it contains no subgraph homeomorphic to K_4. Let $\mathcal{F}$ be the set of graphs that do not contain a K_4 minor. Throughout this section we suppose G is a smallest member of $\mathcal{F}$ which is not a maximum-demand graph, that is, G is not a maximum-demand graph but every proper subgraph of G is. Note that $es(G)$ and $\theta_v(G)$ are the sums of the corresponding values of the components of G. Thus, since G is minimum in $\mathcal{F}$, G must be connected. Suppose G has a cutvertex. Then there exist nonempty subgraphs H and K of G such that $G = H \circ K$. Since G is minimum in $\mathcal{F}$; H, K, and all subgraphs of H and K are maximum-demand graphs. Thus, by Proposition 10, G is a maximum-demand graph, a contradiction. Thus, we may assume that G is 2-connected.

Lemma 17 *Suppose G is 2-connected and does not contain K_4 as a minor. Then either G is a cycle or G contains a cycle with exactly two vertices of degree greater than two.*

Proof: If G is not a cycle, then obtain G' from G by suppressing all divalent vertices. Note that G' is still 2-connected and does not contain K_4 as a minor. Therefore G' is a series-parallel multigraph and can be constructed recursively from a K_2 by the operations of subdividing and doubling edges (see Diestel [3], p. 185). By construction G' contains no divalent vertices; hence, G' must contain parallel edges. Any two parallel edges in G' correspond to a cycle in G which satisfies the conclusion of the Lemma. □

Theorem 18 *Every graph which does not contain a K_4 minor is a maximum-demand graph.*

Proof: By the remarks preceding Lemma 17, G is 2-connected. If G is a cycle, we are done. Otherwise let G be a 2-connected minimum member of $\mathcal{F}$ with respect to being a maximum-demand graph. By Lemma 17, G contains a cycle C with exactly two vertices, x and y, of degree greater than two. Let P_1 and P_2 be the two internally disjoint paths in C with ends x and y. Let l_i be the length of P_i and assume, without loss of generality, that $l_1 \geq l_2$. Let H be the subgraph of G which is obtained by deleting the edges and internal vertices of P_1. Since G is minimum in $\mathcal{F}$, H and all of its subraphs are maximum-demand graphs. If $l_1 \geq 3$, Theorem 15 implies G is a maximum-demand graph, a contradiction. If $l_1 = 2$ and $l_2 = 1$, Theorem 16 shows

G is a maximum-demand graph, again a contradiction. If $l_1 = 1$, then P_1 and P_2 are parallel edges which implies $\theta_v(G) = \theta_v(H)$ and $es(G) = es(H)$. Since H is a maximum-demand graph, so is G, yet another contradiction.

The only remaining case to consider is $l_1 = l_2 = 2$. Let $V(P_1) = \{x, u, y\}$ and $V(P_2) = \{x, v, y\}$. Let S be an es-set for G such that $S \cap V(H - C)$ is maximized. Obtain an es-set S' for G from S by attacking u and v. These attacks cannot be responded to by vertices in $H - C$. Thus $S' \cap V(H-C) = S \cap V(H-C)$. By Lemma 11 $S' \cap V(H-C)$ is an eternally secure set for $H-C$. Thus, $es(G) = es(S') \geq es(H-C)+2$. Adding $\{x, u\}, \{y, v\}$ to any clique cover of $H - C$ produces a clique cover of G, so $\theta_v(G) \leq \theta_v(H - C) + 2$. Since G is minimum in $\mathcal{F}$, $es(H - C) = \theta_v(H - C)$. Therefore, $\theta_v(G) \leq es(H - C) + 2 \leq es(G)$ which implies $\theta_v(G) = es(G)$, once again contradicting the assumption that G is not a maximum-demand graph. The theorem now follows. □

The following corollary is immediate.

Corollary 19 *Every series-parallel graph is a maximum-demand graph.*

6 Open Questions

The study of eternal security seems to be both difficult and rife with interesting questions. Here are three.

1. Goddard et al [4] wonder if toroidal graphs are clique-preserving. We take this down a notch and pose the same question for planar graphs.

2. If S is an eternally secure set, does S contain a minimum eternally secure subset?

3. Goddard et al [4] show that if $\beta_v(G) = 2$, then $es(G) \leq 3$ and they ask if there is a similar bound on $es(G)$ when $\beta_v(G) = 3$. We feel this question is worth repeating and offer the following partial solution: If $\beta_v(G) = 3$ and there exists an independent set of vertices $S = \{a_1, a_2, a_3\}$ which is in no induced $K_{3,3}$, then $es(G) \leq 15$. To see this let A_i be the set of vertices adjacent to a_i. Notice that $\{a_i\} \cup (A_i - (A_j \cup A_k))$ is complete and $\beta_v((A_i \cap A_j) - A_k) \leq 2$ for distinct i, j, and k. Furthermore, since S is in no induced $K_{3,3}$, $\beta_v(A_i \cap A_j \cap A_k) \leq 2$. By the Goddard et al result, $es((A_i \cap A_j) - A_k) \leq 3$ (three sets) and $es(A_i \cap A_j \cap A_k) \leq 3$. Adding one guard for each complete graph $\{a_i\} \cup (A_i - (A_j \cup A_k))$ yields the 15.

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