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Graceful graphs with pendant edges

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Abstract

A graceful labeling of a graph G of size n is an injective assignment of labels from $\{0, 1, ..., n\}$ to the vertices of G such that when each edge of G has assigned a weight defined by the absolute difference of its end-vertices, the resulting weights are distinct. In this paper, we study graceful labelings of graphs with pendant edges attached; in particular, we provide graceful labelings for graphs of the form $G \odot nK_1$ and $G + nK_1$ when G is a graceful graph with order greater than its size. We also show a graceful labeling of the unicyclic graph formed by a cycle with any number of pendant edges attached.

1 Introduction

A function f is a *graceful labeling* of a graph G of size n if f is an injection from $V(G)$ to the set $\{0, 1, ..., n\}$ such that, when each edge xy of G has assigned the *weight* $|f(x) - f(y)|$, the resulting weights are distinct; in other words, the set of weights is $\{1, 2, ..., n\}$. A graph that admits a graceful labeling is said *graceful.* The notion of graceful labeling was introduced by Rosa in 1966 [15] with the name of *β-valuation*, as a tool for decomposing the complete graph into isomorphic subgraphs. The interested reader is referred to Bosák's monograph [2]. Graph labelings can also be applied in areas such coding theory, communication networks, mobile telecommunications or optimal circuits layouts.

Many of the results about graph labeling are collected and updated regularly in a survey by Gallian [7]. The reader can consult this survey for more information about the subject. The notation and terminology used in this paper are taken from [7].

In 1984, Truszczýnski [18] conjectured that all unicyclic graphs (i.e., graphs with a unique cycle) except the cycle C_m, where $m \equiv 1$ or 2(mod4), are graceful. Some results support this conjecture. Truszczyński proved that dragons are graceful, a *dragon* being the graph obtained by joining an end-vertex of a path to a cycle. He proved that the one point union of a graceful cycle C_m with any tree that admits an α-labeling, results in a graceful graph. Cycles with pendant edges attached have been studied. Frucht [5] proved that the corona $C_m \odot K_1$, i.e., the cycle C_m with a pendant edge attached at each vertex is graceful (the corona operation between two graphs is formally introduced in the next section.) Frucht's result was generalized independently by the author [1] and Bu, Zhang and He [3]; thus, the corona $C_m \odot nK_1$ is graceful for every $n \geq 1$ and $m \geq 3$. Ropp [15] proved that the graphs obtained by attaching one pendant point to each vertex of a prism, i.e., $(C_m \times P_2) \odot K_1$, or by attaching one pendant point to each vertex of one cycle of a prism are graceful. Sethuraman and Elumalai [17] proved that $K_{m,n} \odot K_1$ is graceful for m even and $m \leq n \leq 2m+4$ and for m odd and $m \leq n \leq 2m-1$. They also proved that $K_{1,r,s} \odot K_1$ is graceful for all positive integers r, s. The methods to construct new graceful trees, given in 1973 by Stanton and Zarnke [17], in 1979 by Koh and Tan [13], and in 1983 by Grace [9], can be used to prove that the corona $T \odot nK_1$ (i.e., a tree with n pendant edges attached to each vertex) is graceful when T is a graceful tree. Motivated by these examples we explore a more general case related to the corona operation of a graph and nK_1. The results obtained are included in Section 2.

Let C_m^t denote the class of graphs formed by adding a single pendant edge to t vertices of the cycle C_m, where $1 \leq t \leq m$. Ropp and Gallian [6] conjectured that for all m and t, all members of this class are graceful. This conjecture was proved by Kang, Liang, Gao, and Yang [12]. In Section 3 we prove a stronger result, namely, we prove that any cycle with any number of pendant edges attached to its vertices is graceful; thus coming closer to the conjecture of the gracefulness of unicyclic graphs posed by Truszczyński [18].

2 Graphs with Pendant Edges Attached

In 1970, Frucht and Harary [4] presented a new operation between two graphs, namely, the corona of two graphs. Given two graphs G and H, the *corona* of G with H, denoted by $G \odot H$, is the graph with $V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i)$, and $E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} E(H_i) \cup \{(i, u_i) : i \in V(G) \text{ and } u_i \in V(H_i)\}$.

In other words, a corona graph is obtained from two graphs, G and H, taking one copy of G, which is supposed to have order p, and p copies of H, and then joining by an edge the k^{th} vertex of G to every vertex in the k^{th} copy of H.

Given a graceful graph G, is $G \odot nK_1$ graceful for some value of n? Under certain constraints, we can answer affirmatively to this question: if G is a graceful graph of order m and size $m-1$, the corona $G \odot nK_1$ is graceful for every value of n. We also study here coronas of the form $K_1 \odot G$, and prove that these graphs are graceful when G is a graceful graph of order m and size $m-1$.

The following results show a pair of procedures to create graceful graphs using the corona operation as well as the join of two graphs.

Theorem 1 *Let G be a graceful graph of order m and size $m-1$. The corona $G \odot nK_1$ is graceful.*

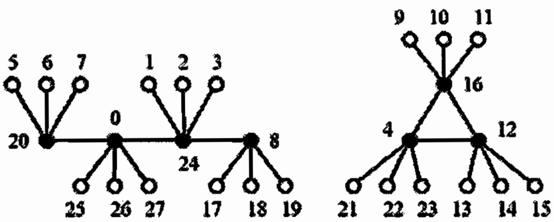

Figure 1: Graceful corona of $C_3 \cup P_4$ and $3K_1$

Proof. The graph $G \odot nK_1$ has order $mn + m$ and size $mn + m - 1$. If f is a graceful labeling of G, f assigns the labels $0, 1, ..., m-1$ on the m vertices of G and induces the weights $1, ..., m-1$. Assume that the vertices of G in $G \odot nK_1$ have been labeled using the labeling f. Let us multiply these labels by $(n+1)$; thus, we have used the integers $0, (n+1), ..., (m-1)(n+1)$ as labels, producing the weights $(n+1), 2(n+1), ..., (m-1)(n+1)$. If a vertex v of the copy of G has label $t(n+1)$, the n end-vertices adjacent to v are labeled with consecutive integers where the greatest integer used is $mn + m - 1 - t(n+1)$. Thus, the weights of the n pendant edges attached to v are n consecutive integers not multiples of $n+1$. Clearly , all the weights on the pendant edges are different between them and different from the weights on the edges of G. Since the labels are $0, 1, ..., mn + m - 1$ and each induced weight appears exactly once, we have $G \odot nK_1$ gracefully labeled. □

In Figure 1 we show an example of this construction for $(C_3 \cup P_4) \odot 3K_1$. Dividing the label of the black vertices by 4, we recover the graceful labeling of $C_3 \cup P_4$.

The constraint on the size of G, in the above theorem, can be removed. Deleting this condition, we are able to prove that $G \odot nK_1$ is an induced subgraph of a graceful graph H.

Let G be a graceful graph of order p and size q, with $q > p$. Denote by G' the graph $G \cup (q+1-p)K_1$; note that G' has order $q+1$ and size q. Thus, using the result of Theorem 1 we can prove that $H = G' \odot nK_1$ is a graceful graph.

Theorem 2 *Let G be a graceful graph of order p and size q, with $q > p$. Let $G' = G \cup (q+1-p)K_1$, then $G' \odot nK_1$ is also a graceful graph.*

Proof. Let f be a graceful labeling of G, then f assigns to the p vertices of G numbers from $\{0, 1, ..., q\}$. Thus, there are $q+1-p$ numbers in this set that have not been assigned. Therefore, assigning them to the vertices of $(q+1-p)K_1$ in G', we obtain a graceful labeling of G'. Since G' is graceful and has order $q+1$ and size q, we can apply Theorem 1 to prove that $G' \odot nK_1$ is graceful. □

Consider now the join of two graphs $G + H$, when $H = K_1$ this join is the corona $K_1 \odot G$. Graham and Sloane [8] showed a graceful labeling of the join of the path P_n and K_1, i.e., the *fan* (or *shell*) $F_n = P_n + K_1$, while Hebbare [11] proved that the join of the star S_n and K_1, $S_n + K_1$, is graceful for every n. Later, Grace [10] generalized both results showing that if T is any graceful tree, then $T+nK_1$ is also graceful. Below we provide a generalization of Grace's result.

Theorem 3 *Let G be a graceful graph of order m and size $m-1$. The join of G and nK_1 results in a graceful graph.*

Proof. The graph $G+nK_1$ has order $m+n$ and size $m(n+1)-1$. If f is a graceful labeling of G, then f assigns the labels $0,1,...,m-1$ to the vertices of G. Let $g: V(G+nK_1) \to \{0,1,...,m(n+1)-1\}$, such that g assigns the labels $0,1,...,n-1$ to the n vertices of nK_1 and $g(v)=(n+1)f(v)+n$ for each $v \in V(G)$.

We claim that g is a graceful labeling of $G+nK_1$. In fact, note that the labels assigned for g on the vertices of G are $n, 2n+1, 3n+2, ..., mn+(m-1)$. The weights induced on the edges of G are $(n+1), 2(n+1), ..., (m-1)(n+1)$. Since the labels $0,1,...,n-1$ are assigned to the vertices of nK_1 and all of them are connected to each vertex of G, the weights of the edges connecting them with the vertex of G with label $tn+(t-1)$, where $1 \leq t \leq m$, are $tn+t-1, tn+t-2, ..., tn+t-n$. Thus,

t	induced weights
1	$1,2,...,n$
2	$n+2, n+3, ..., 2n+1$
$\vdots$	$\vdots$
m	$m(n+1)-n, m(n+1)-n+1, ..., m(n+1)-1.$

In conclusion, g assigns the labels $0,1,...,n-1$ and $n, 2n+1, 3n+2, ..., mn+(m-1)$, and induces the weights $1,2,...,m(n+1)-1$. Therefore, g is a graceful labeling of $G+nK_1$. □

In Figure 2 we show an example of this labeling for $(C_5 \cup P_3)+2K_1$. In particular, when $n=1$ in the above theorem, we have proved that the corona $K_1 \odot G$ is graceful when G is a graceful graph of order m and size $m-1$. Theorem 3 can be extended to any graceful graph in the same way that Theorem 1 was extended. Thus, we have that, given any graceful graph G, $G'+nK_1$ is also graceful.

3 Graceful Hairy Cycles

A unicyclic graph G, other than a cycle, is called a *hairy cycle* if the deletion of any edge e in the cycle of G results in a caterpillar. Thus, we can see the coronas $C_n \odot mK_1$ are examples of hairy cycles. In this section we prove that all hairy cycles are graceful. One of the tools used to prove this statement is the graceful labeling of caterpillars given by Rosa [15]. We present this labeling in the same way used by Rosa in his pioneering article, that is, using a figure. Here, a bipartition of the vertices of the caterpillar is chosen, such that the vertices are arranged in the order of the underlaying path, and labels are applied as shown in Figure 3.

Theorem 4 *All hairy cycles are graceful.*

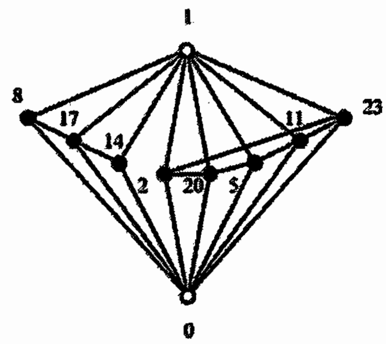

Figure 2: Graceful labeling of $(C_5 \cup P_3) + 2K_1$

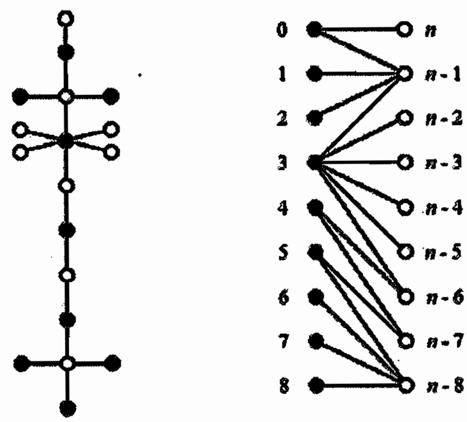

Figure 3: Graceful labeling of caterpillars

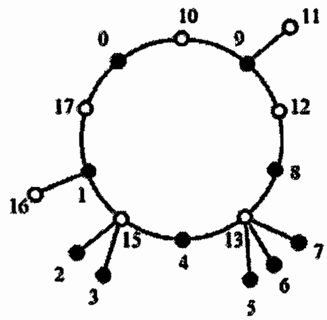

Figure 4: Graceful hairy cycle

Proof. Let G be a hairy cycle of size m and girth n. Let $e = uv$ be an edge in the unique cycle of G. Then $G - e$ is a caterpillar. Let $\{A, B\}$ be the bipartition of the vertices of $G - e$ with $u \in A$. Let f be a Rosa's α-labeling of the caterpillar $G - e$. Without loss of generality we may assume that $f(u) = 0$ and $f(v) = k$, where $k = |A| - 1 = \max f(A)$ when n is odd and $k = |A| = \min f(B)$ when n is even. We identify the vertices of $G - e$ with the labels assigned to them by f. Let ab be the edge with weight k, $a \in A$ and $b \in B$. The labeling f is said to be *good* if a is the vertex with largest label among the neighbors of b (the edge ab has the smallest weight among the edges incident with b.) Otherwise f is said to be *bad*. We consider two cases.

Case 1: f is good. Then consider the following labeling g of G :

$$g(x) = \begin{cases} f(x), & \text{if } x \in A \\ f(x), & \text{if } x \in B \text{ and } x < b \\ f(x) + 1, & \text{if } x \in B \text{ and } x \geq b. \end{cases}$$

We claim that g is a graceful labeling of G. In fact, the edges of $G - e$ with weight at most $k - 1$, keep their weights in G, the edge e has weight k, and the edges of $G - e$ with weight at least k have increased their weights in one unit in G; so, these weights are $k + 1, \ldots, m$. Therefore, g is a graceful labeling of G. In Figure 4 we show an example, where the edge e has end vertices 0 and 10.

Case 2: Suppose now that, for each edge $e = uv$ in the cycle of G, all α-labelings of $G - e$ of Rosa's type are bad. This case is subdivided into two subcases.

Case 2.1: There is vertex v in the cycle of G with degree 2. Since G is different from a cycle, we may assume that there is a neighbor u' of v with degree $d(u') > 2$. Let $e = uv$, where $u \neq u'$ is the other neighbor of v and consider $G - e$. As before, denote by ab the edge of weight k in an α-labeling f of $G - e$ with $f(u) = 0$ and $f(v) = k$. Since f is bad, there is a neighbor of b in $G - e$ with a label larger than a. Therefore, the end vertices of the edge with weight $k + 1$ are $a'b'$ with $b' > b$ and $a' \leq a$, and now, by the structure of the α-labelings of Rosa's type, a' has the largest label among the neighbors of b'. On the other hand, since v has degree 1 in $G - e$, there is an end vertex v' adjacent to u' with label $k + 1$. Note that G is also

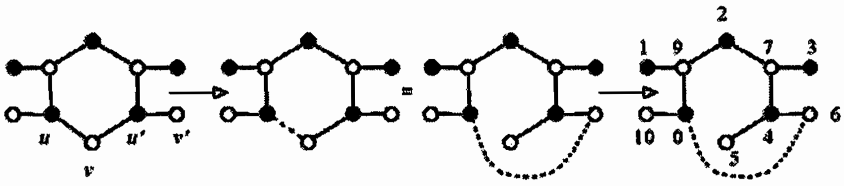

Figure 5: Graceful hairy cycle

isomorphic to $G - e + uv'$.

Thus, we are able to define a graceful labeling g of G as follows:

$$g(x) = \begin{cases} f(x), & \text{if } x \in A \\ f(x), & \text{if } x \in B \text{ and } x < b' \\ f(x) + 1, & \text{if } x \in B \text{ and } x \geq b'. \end{cases}$$

The edges of $G - e$ with weight at most k keep their weights in G. The edges of $G - e$ with weight at least $k + 1$ have increased their weights exactly one unit on G. Finally, the edge uv' (used in place of uv to obtain G) has weight $k + 1$. Then, g is a graceful labeling of G. We show an example in Figure 5.

Case 2.2: Suppose that every vertex on the cycle has degree greater than 2. Let ab be the edge of $G - e$ with weight k. Now, we proceed to describe a new labeling g of G based in the labeling f of $G - e$.

When $x \in A$, $g(x) = f(x)$ if $x \leq a - 2$ or $x > a$, $g(a - 1) = a$, and $g(a) = a - 1$.

When $x \in B$, $g(x) = f(x)$ if $x \leq b$ or x is an end vertex of G adjacent to a; otherwise, $g(x) = f(x) + 1$.

Next we check that g is in fact a graceful labeling of G. Let $t = \deg(a) - 2$.

- The edges xy with $x > a$ and $y \leq b$ have the same weight in both $G - e$ and G. These weights are $1, 2, \ldots, k - 1$.

- The edge $e = uv$ of G has weight k.

- The edge ab of $G - e$, with weight k is transformed into the edge $(a - 1)b$ in G, its new weight is $k + 1$.

- The edges ay in $G - e$, where either $y > b$ or $d_{G-e}(y) = 1$, whose weights were $k+1, \ldots, k+t$ are transformed into the edges $(a-1)y$ in G, being $k+2, \ldots, k+t+1$ the corresponding new weights.

- The edge ay in $G - e$, whose weight was $k + t + 1$, since $y > b$ and $d_{G-e}(y) > 1$, is transformed into the edge $(a - 1)(y + 1)$ in G, whose weight is $k + t + 3$.

- The edge $(a - 1)y$ in $G - e$, whose weight is $k + t + 2$, is transformed into the edge $a(y + 1)$ in G, keeping its weight $k + t + 2$.

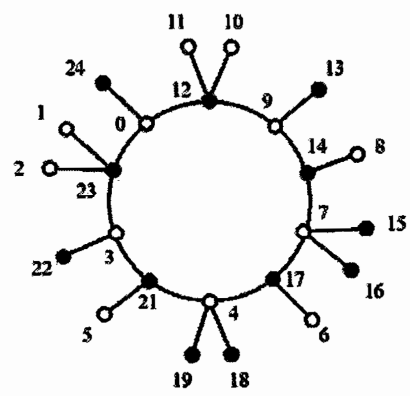

Figure 6: Graceful hairy cycle

- The edges xy in $G-e$, where $x \leq a-2$, whose weights were $k+t+2,\ldots,m-1$, are transformed into the edges $x(y+1)$ in G, increasing their weights one unit, becoming the weights $k+t+3,\ldots,m$ in G.

Thus, g is effectively a graceful labeling of G. We show an example of this case in Figure 6. □

Note that when the girth n is an even number, the labeling g is an α-labeling. That is, g is a graceful labeling with the additional property that there exists a number λ so that for each edge xy either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$. The value of λ is the smaller of the two labels that yield the edge with weight 1.

4 Conclusions

The attachment of pendant edges to a graceful graph seems to be a useful tool to generate new graceful graphs. Noting that the same idea works over graphs that are not graceful, producing graceful graphs anyway, we can consider the case of the complete graph K_5 or the cycle C_m for $m \equiv 1$ or $2(\text{mod}4)$.

It seems that the idea of Theorem 4 can be extended to unicyclic graphs (different from a cycle), such that the deletion of one edge of the cycle produces a tree with an α-labeling. Unfortunately, trees that admit α-labelings have not been characterized yet; however, the theorem suggests deeper explorations.

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