Graceful labelings of chain and corona graphs

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Abstract

In this paper we study graceful labelings of some graphs that result of two different constructions. The first construction produces chain graphs, i.e., a concatenation of graphs. A chain graph will be graceful if any graph in the chain, accepts an α-labeling. The second construction is the corona product; graceful labelings of two families of corona graphs are obtained. The results presented here, generalize some former labelings.

1 Introduction

Graph theory terminology and notation are taken from [4]. A *labeling* f of a graph G is a one-to-one mapping from the vertex set of G into a set of integers. For each edge $e = uv \in E(G)$, the *weight* induced by f on e is the number $|f(u) - f(v)|$. Let G be a graph of order n and size m, if $f : V(G) \to \{0, 1, ..., m\}$ is a labeling of G, such that the set of weights induced by f is $\{1, 2, ..., m\}$, f is said a *graceful labeling* of G, and G is called a *graceful graph.* If we replace each vertex label $f(v)$ by $m - f(v)$, then we have a new graceful labeling of G, called the *complementary labeling* of f, and denoted $\overline{f}$. Note that both labelings induce the same weights. Graceful labelings were introduced in 1967 by Rosa [11], who also defined an α - *labeling* as a graceful labeling f with the additional property, that there exists an integer λ (called the *boundary value* of f,) so that for each edge xy, either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$. Two conclusions follows from the definition: first, λ must be the smallest of the two vertex labels that yield the edge of weight 1, and second, a graph with an α-labeling is necessarily bipartite.

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More information about this topic may be founded in the last version of the survey written by Gallian [7].

In 1988, Rosa [12] defined a *triangular snake* as a connected graph in wich all blocks are triangles and the block-cutpoint graph is an snake (or path). He also conjectured that Δ_n-snake (i.e., a triangular snake with n blocks) is graceful for $n \equiv 0$ or 3(mod 4). A natural generalization of that definition is the following. An *mG-snake* is a connected graph (with m blocks) in which all blocks are isomorphic to a graph G (which is supposed to be a block) and the block-cutpoint graph is an snake. Moreover, instead of take m copies of graph G, we take m graphs, not necessarily isomorphics, and connect them in such a way that the block-cutpoint graph is an snake. The graph so constructed will be a chain graph. In Section 2, we give some general results about α-labelings of this kind of graphs.

The following contruction of graphs is due to Frucht and Harary [5], who introduced it in 1970. Given two graphs G and H, the *corona* (crown) of G with H, denoted by $G \odot H$, is the graph with

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$

$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} E(H_i) \cup \{(i, u_i) : i \in V(G) \text{ and } u_i \in V(H_i)\}.$$

In other words, a corona graph is obtained from two graphs, G and H, taking one copy of G, which is supposed to have order p, and p copies of H, and then joining by an edge the k^{th} vertex of G to every vertex in the k $^{\mathrm{th}}$ copy of H. In Section 3, we give a graceful labeling of the coronas, where G is a cycle and H is mK_1. We also observe that the corona $K_n \odot K_1$ is graceful.

2 Graceful labelings of chain graphs

As we said before, if G is a graph that is a block, an *mG-snake* is a connected graph in which the m blocks are isomorphic to G and whose block-cutpoint graph is an snake. When G is any cycle, we said that *mG-snake* is a *cyclic snake*. The gracefulness of some of these graphs has been studied. In 1989, Moulton [10] gave graceful labelings of *mC_3-snakes* when $n \equiv 0$ or 1(mod 4) , answering the conjecture of Rosa, that has been mentioned above. Ruiz [13] in 1979 and Gnanajothi [8] in 1991, worked *mC_4-snakes*, giving graceful labelings of some particular cases. Recently, the author [3] has gaven a labeling for the general case of these cyclic snakes and for other ones, where the cycles are C_6 or cycles of size multiple of 4.

This result, answers in part a conjecture posed by Acharya in 1983 [1] about the k -gracefulness of d-dimensional polyominoes.

Suppose now, that the graphs $B_1, B_2, ..., B_m$ are blocks and that for every $i \in \{1, 2, ..., m-1\}$, B_i and B_{i+1} have a vertex in common, in such a way that the block-cutpoint graph is a path. The graph G obtained of this concatenation will be called a *chain graph* . This definition generalize the concept of *mG-snakes*. Our next theorem, establish that when every block B_i accepts an α -labeling, it is possible to construct a chain graph, whose blocks are the B_i, that also has an α-labeling. But before that, we will give the definition of a *k-graceful labeling*, that was introduced independently in 1982 by Slater [14] and by Maheo and Thuillier [9] , and connect this one with the α-labelings.

A graph G with q edges is *k-graceful* if there is a labeling f from $V(G)$ to $\{0, 1, ..., q+k-1\}$ such that the weights induced are $k, k+1, ..., q+k-1$. The numbering f is called a k *-graceful labeling*. Any graph that has an α-labeling is k -graceful for all k. In fact, let $G = (X \cup Y, E)$ be a bipartite graph with an α-labeling f, then f assigns the integers $0, 1, ..., |X|-1$ to the vertices of X, the labels of Y are in the set $\{|X|, ..., |E|\}$. In order to obtain a k-graceful labeling of G, is suffices add to each label of f a non-negative constant t $(= k-1)$. Now we present the theorem.

Theorem 1. *Let $B_1, B_2, ..., B_m$ be blocks such that all of them have an α -labeling. Then, there exists a chain graph G, with blocks $B_1, B_2, ..., B_m$ that accepts an α-labeling.*

Proof. Denote by q_i the size of B_i, by f_i the α-labeling of B_i and let X_i and Y_i the bipartite sets of B_i. Without loss of generality, we may assume that f_i assigns the integers $0, 1, ..., |X_i|-1$ to the vertices of X_i. For every $1 \leq i \leq m-1$, identify the vertex of B_i with the label $|X_i|-1$ with the vertex of B_{i+1} with label 0; the graph so obtained, is a chain graph, denoted by G. The order of G is $1 - m + \sum_{i=1}^{m} |X_i| + |Y_i|$, its size is $q = \sum_{i=1}^{m} q_i$, and its bipartite sets are $X = \bigcup_{i=1}^{m} X_i$ and $Y = \bigcup_{i=1}^{m} Y_i$. Note that all the cut-vertices of G are in X, so $|X| = \sum_{i=1}^{m} |X_i| - m + 1$.

Now we shall obtain the α-labeling of G. Denote by $\overline{q}_i$ the sum $q_{i+1} + ... + q_m + 1$ for $1 \leq i \leq m-1$ and let $\overline{q}_m = 1$. Adding the suitable constant $\overline{q}_i - 1$, we transform f_i into the $\overline{q}_i$-graceful labeling g_i. Then the weights induced on B_i by g_i are $\overline{q}_i, \overline{q}_i + 1, ..., \overline{q}_i + q_i - 1$. Therefore, the weights on G are:

$$
\begin{array}{l}
1, 2, ..., q_m, \\
q_m + 1, q_m + 2, ..., q_m + q_{m-1}, \\
q_{m-1} + q_m + 1, q_{m-1} + q_m + 2, ..., q_m + q_{m-1} + q_{m-2}, \\
\vdots \\
q_2 + ... + q_m + 1, q_2 + ... + q_m + 2, ..., q_m + q_{m-1} + ... + q_1,
\end{array}
$$

i.e., $1, 2, ..., q$.

Now, in order to have the graceful labeling f, we need to shift, conveniently, the labelings just obtained. Add to the labels of B_i, for $i \geq 2$, the constant $|X_{i-1}|$. Then, B_{i-1} and B_i has a vertex with label $|X_{i-1}|$, that will be the cut-vertex between them. Since the labels on G are in the set $0, 1, ..., q$, the labeling obtained is a graceful labeling f.

The boundary value λ of f is obtained on B_m and correspond to the smallest label on the extrems of the edge with weight 1. Therefore, f is an α-labeling of G. □

In Figure 1 we show an example of this labeling.

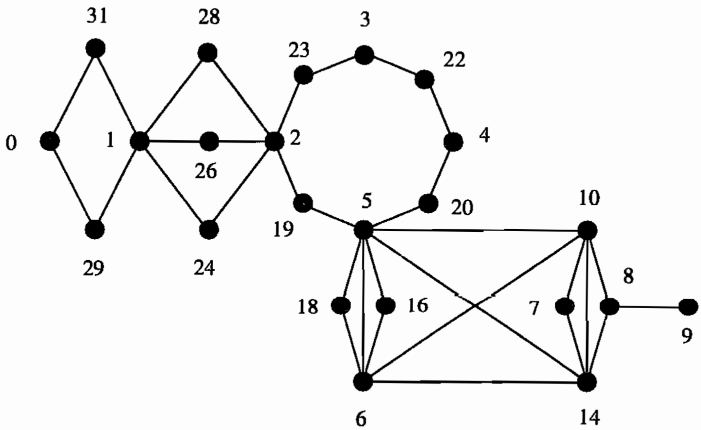

Figure 1

Let G be a chain graph with m blocks ($m \geq 3$). Let u_i be the vertex that connect B_i with B_{i+1}, $1 \leq i \leq m-1$. So, $\{u_1, u_2, ..., u_{m-1}\}$ is the set of cut-vertices of G. Let $d_i = d(u_i, u_{i+1})$, namely, the distance between u_i and u_{i+1}, $1 \leq i \leq m-2$. Therefore, each chain graph G has associated the string $d_1, d_2, ..., d_{m-2}$. When every block is a vertex-transitive graph, G may be described by its string. The strings of the chain graphs obtained following the construction of Theorem 1, are conditioned for the distance between the vertices labeled with 0 and q_i, on every block B_i.

Let G be any chain graph with m blocks $B_1, B_2, ..., B_m$, such that B_i is any complete bipartite graph. Then G has an string $d_1, d_2, ..., d_{m-2}$, where $d_i \in \{1, 2\}$. In this case is possible to obtain an α-labeling of G, using the idea of construction in the previous theorem and the α-labeling of $K_{m,n}$ gaven by Rosa [11]. In the next lemma, we present that labeling of $K_{m,n}$, which will be used in the theorem below.

Lemma 1. *The graph $K_{m,n}$ has an α-labeling.*

Proof. Assume that $K_{m,n}$ has bipartite sets M and N, such that $|M| = m$ and $|N| = n$. The vertices of M are labeled with the integers $0, 1, ..., m-1$. The vertices of N are labeled with the numbers $m, 2m, ..., nm$. The constant λ is $m-1$. □

Our next theorem, generalize the graceful labelings of mC_4-snakes obtained by the author in [3], where the blocks were isomorphic to $K_{2,2}$.

Theorem 2. *Let $B_1, B_2, ..., B_m$ be complete bipartite graph, Then any chain graph G, obtained by the concatenation of these blocks, has an α -labeling.*

Proof. Let $d_1, d_2, ..., d_{m-2}$ the string associated to G. Let $B_i = K_{m_i,n_i}$ with bipartite sets M_i and N_i of order m_i and n_i, respectively. In the construction of the desired labeling, the sets M_i always have the smallest labels of B_i. Without loss of generality, we may assume that the cut-vertex between B_1 and B_2 belongs to M_1 and M_2.

The vertices of M_1 are labeled with the integers $0, 1, ..., m_1 - 1 = a_1$, and the vertices of N_1 are labeled with the integers $q, q-m_1, q-2m_1, ..., q-(n_1-1)m_1 = q_1$, where $q = \sum_{i=1}^{k} m_i n_i$.

The vertices of M_2 are labeled with the integers a_1,$a_1+1, ..., a_1+m_2-1 = a_2$, and the vertices of N_2 are labeled with the integers $q_1-1, q_1-1-m_2, q_1-1-2m_2, ..., q_1-1-(n_2-1)m_2 = q_2$.

When $d_i = 2$, the vertices of M_{i+2} are labeled with the integers a_{i+1}, $a_{i+1}+1, ..., a_{i+1}+m_{i+2}-1 = a_{i+2}$, and the vertices of N_{i+2} are labeled with the integers $q_{i+1}-1, q_{i+1}-1-m_{i+2}, q_{i+1}-1+2m_{i+2}, ..., q_{i+1}-1-(n_{i+2}-1)m_{i+2} = q_{i+2}$.

When $d_i = 1$, the vertices of M_{i+2} are labeled with the integers $a_{i+1}+1$, $a_{i+1}+1+n_{i+2}, a_{i+1}+1+2n_{i+2}, ..., a_{i+1}+1+(m_{i+2}-1)n_{i+2} = a_{i+2}$, and the vertices of N_{i+2} are labeled with the integers $q_{i+1}, q_{i+1}-1, ..., q_{i+1}-n_{i+2}+1 = q_{i+2}$.

Note that for every block B_i ($i \geq 2$), the numbering obtained is a d-graceful labeling shifted conveniently. For B_1 the numbering is just a d-graceful. It can be checked that the entire labeling is graceful and looking on B_k, for the edge of weight 1, the value of its smaller extreme corresponds to the boundary value of the labeling. This conclude the proof. □

In Figure 2, we show an example of this construction.

3 Graceful labelings of corona graphs

An special kind of corona graph is given by $C_n \odot K_1$, i.e., a cycle with pendant points. In 1979 Frucht [6] shown that this corona is graceful.

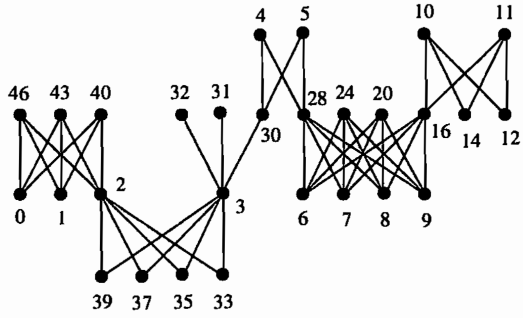

Figure 2

Following Frucht's idea, we studied the gracefulness of the corona $C_n \odot mK_1$, obtaining that any cycle with m pendant points attached ($m \geq 1$) is graceful. To avoid an excessive number of figures, we represent the labelings of $C_n \odot mK_1$ in a rectangular array of n columns and $m+1$ rows. The first row containing the labels of C_n; from the second position of any column, are the labels of the end-vertices, as can be seen from Figure 3.

$$C_8 \odot 2K_1 : \begin{cases} 0, & 22, & 3, & 19, & 7, & 16, & 10, & 13 \\ 23, & 1, & 20, & 4, & 17, & 8, & 14, & 11 \\ 24, & 2, & 21, & 5, & 18, & 9, & 15, & 12 \end{cases}$$

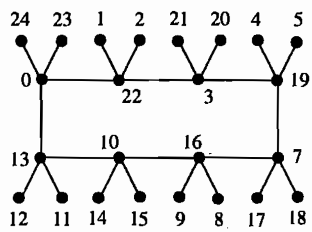

Figure 3

Theorem 3. *The corona $C_n \odot mK_1$ is a graceful graph for every positive integers $n \geq 3$ and $m \geq 1$.*

Proof. Describe the cycle by the circuit $v_1, v_2, ..., v_n, v_1$ and denote by $v_{i,j}$ the vertices of degree one adjacents to v_i ($1 \leq i \leq n$ and $1 \leq j \leq m$.) We distinguish four cases:

(i) When $n \equiv 0 (\text{mod } 4)$ the labeling f defined below is a graceful labeling of $C_n \odot mK_1$.

$$f(v_{2k+1}) = \begin{cases} (m+1)k, & 0 \leq k \leq \frac{n-4}{4} \\ (m+1)k+1, & \frac{n}{4} \leq k \leq n-1 \end{cases}$$

$$f(v_{2k}) = (m+1)(n-k)+1, \ 1 \leq k \leq \frac{n}{2}$$

and

$$f(v_{i,j}) = \begin{cases} (m+1)n - f(v_i) - j + 1, & 1 \leq i \leq \frac{n}{2}, \ 1 \leq j \leq m \\ (m+1)n - f(v_i) - j + 2, & \frac{n}{2} + 1 \leq i \leq n, \ 1 \leq j \leq m \end{cases}$$

$$\text{Example, } C_4 \odot 3K_1 : \begin{cases} 0, 13, \ 5, \ 9 \\ 14, \ 1, 10, \ 6 \\ 15, \ 2, 11, \ 7 \\ 16, \ 3, 12, \ 8 \end{cases}$$

(ii) When $n \equiv 1 (\text{mod } 4)$ the labeling f defined below is a graceful numbering of $C_n \odot mK_1$.

$$f(v_{2k+1}) = \begin{cases} (m+1)k, & 0 \leq k \leq \frac{n-1}{2}, \ k \neq \frac{n-1}{4} \\ (m+1)k-1, & k = \frac{n-1}{4} \end{cases}$$

$$f(v_{2k}) = \begin{cases} (m+1)(n-k)+1, & 1 \leq k \leq \frac{n-1}{4} \\ (m+1)(n-k), & \frac{n+3}{4} \leq k \leq \frac{n-1}{2} \end{cases}$$

and

$$f(v_{i,j}) = \begin{cases} (m+1)n - f(v_i) - j + 1, & 1 \leq i \leq \frac{n-3}{2}, \ 1 \leq j \leq m \\ (m+1)n - f(v_i) + 1, & i = \frac{n-1}{2}, \text{ and } j = 1 \\ (m+1)n - f(v_i) - j + 1, & i = \frac{n-1}{2}, \text{ and } 2 \leq j \leq m \\ (m+1)n - f(v_i) - j - 1, & i = \frac{n+1}{2}, \ 1 \leq j \leq m \\ (m+1)n - f(v_i) - j, & \frac{n+3}{2} \leq i \leq n, \ 1 \leq j \leq m \end{cases}$$

$$\text{Example, } C_5 \odot 3K_1 : \begin{cases} 0, 17, 3, 12, 8 \\ 18, 1, 13, 5, 9 \\ 19, 2, 14, 6, 10 \\ 20, 4, 15, 7, 11 \end{cases}$$

(iii) When $n \equiv 2 \pmod 4$,the labeling f defined below is a graceful labeling of $C_n \odot mK_1$.

$$f(v_{2k+1}) = \begin{cases} (m+1)k, & 0 \le k \le \frac{n-2}{2},\ k \ne \frac{n-2}{4} \\ (m+1)k - 1, & k = \frac{n-2}{4} \end{cases}$$

$$f(v_{2k}) = \begin{cases} (m+1)(n-k)+1, & 1 \le k \le \frac{n-2}{4} \\ (m+1)(n-k), & \frac{n+2}{4} \le k \le \frac{n}{2} \end{cases}$$

and

$$f(v_{i,j}) = \begin{cases} (m+1)n - f(v_i) - j + 1, & 1 \le i \le \frac{n-4}{2},\ 1 \le j \le m \\ (m+1)n - f(v_i) + 1, & i = \frac{n-2}{2}, \text{ and } j = 1 \\ (m+1)n - f(v_i) - j + 1, & i = \frac{n-2}{2}, \text{ and } 2 \le j \le m \\ (m+1)n - f(v_i) - j - 1, & i = \frac{n}{2},\ 1 \le j \le m \\ (m+1)n - f(v_i) - j, & \frac{n+2}{2} \le i \le n,\ 1 \le j \le m \end{cases}$$

$$\text{Example, } C_6 \odot 3K_1 : \begin{cases} 0, 21, 3, 16, 8, 12 \\ 22, 1, 17, 5, 13, 9 \\ 23, 2, 18, 6, 14, 10 \\ 24, 4, 19, 7, 15, 11 \end{cases}$$

(iv) When $n \equiv 3 \pmod 4$, the labeling f defined below is a graceful numbering of $C_n \odot mK_1$.

$$f(v_{2k+1}) = \begin{cases} (m+1)k, & 0 \le k \le \frac{n-3}{4} \\ (m+1)k + 1, & \frac{n+5}{4} \le k \le \frac{n-1}{2} \end{cases}$$

$$f(v_{2k}) = (m+1)(n-k) + 1,\ 1 \le k \le \frac{n-1}{2}$$

and

$$f(v_{i,j}) = \begin{cases} (m+1)n - f(v_i) - j + 1, & 1 \le i \le \frac{n+1}{2},\ 1 \le j \le m \\ (m+1)n - f(v_i) - j + 2, & \frac{n+3}{2} \le i \le n,\ 1 \le j \le m \end{cases}$$

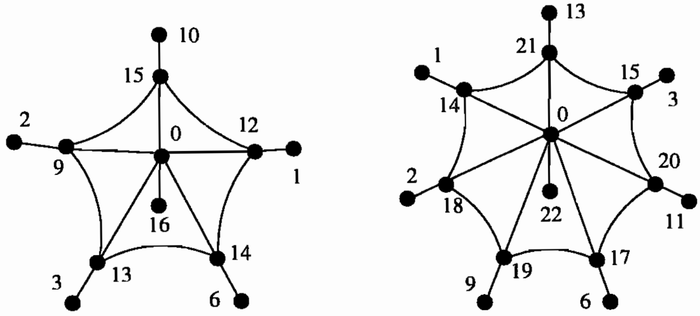

Figure 4

$$\text{Example, } C_3 \odot 3K_1 : \begin{cases} 0,\ 9,\ 5 \\ 10,\ 1,\ 6 \\ 11,\ 2,\ 7 \\ 12,\ 3,\ 8 \end{cases}$$

Note that for n even, the labeling obtained satisfy the additional condition to be an α-labeling.

□

Other corona graph that is graceful is obtained of the wheel W_n and K_1, $W_n \odot K_1$. In 1984, Ayel y Favaron [2], present the graceful numberings of a helm, i.e., a graph obtained from a wheel by adding a pendant vertex on the rim of the wheel, for n even. That graceful labeling assign to the central vertex the label 0. If we attached on it an edge and label the new vertex with the integer $3n + 1$, the resulting graph is $W_n \odot K_1$ with a graceful numbering.

Theorem 4. *The corona graph $W_n \odot K_1$ is graceful, for every $n \geq 3$.*.

In Figure 4, we show two examples for two cases where n is odd.

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