ON 2–EQUITABLE LABELINGS OF GRAPHS

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ABSTRACT. A *labelings* f of a graph G is an assignment of distinct integers to the vertices of G. The *weight* induced by f on each edge $e = uv$ of G is the number $|f(u) - f(v)|$. A labeling of a graph G is said to be k–*equitable* if for each weight h induced by f there are exactly k edges in G having weight h.

In this paper, we present some new results related to k–equitable labelings of graphs. In particular, we give partial characterizations of some 2–equitable graphs. Conjectures and open problems are presented.

1. Introduction

Graph theory terminology and notation are taken from [5]. A *labeling* f of a graph G is a one–to–one mapping from the vertex set of G into a set of integers. For each edge $e = uv \in E(G)$, the *weight* induced by f on e is the number $|f(u) - f(v)|$. A labeling f of G is called *proper* if $f(v) \geq 0$, for every vertex $v \in V(G)$. Note that every labeling f of a graph G can be transformed into a proper labeling of G that preserves the weights induced by f on the edges of G.

A labeling f of a graph G is k–*equitable* if for each weight h induced by f there are exactly k edges of G having weight equal to h. If a graph G has a k–equitable labeling then, G is said to be k–*equitable*. Equitable labelings

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of graphs were introduced by Bloom and Ruiz in [3]. In Figure 1 we show a 3–equitable labeling of the given graph G.

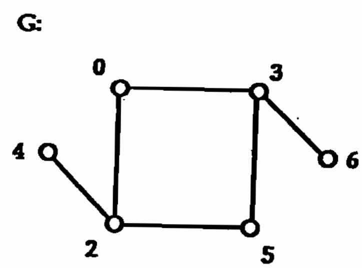


Figure 1

The existence of a k–equitable labeling of a graph G implies the existence of a decomposition of G into linear forests, each of them having size equal to k; namely, the decomposition formed by those subgraphs S_h of G induced by the edges of G that have weight equal to h. It has been shown in [3] that, there exist decompositions of graphs into linear forest of equal size that cannot be obtained from some k–equitable labeling of the corresponding graph. (See also [1].)

Let f be a labeling of a graph G. We say that f is *optimal* if the labels assigned by f to the vertices of G are consecutive integers. Assume that w is the number of distinct weights induced by f on the edges of G. We say that the labeling f is *complete* if the weights induced by f are the integers $1, 2, \dots, w$.

Suppose that G is a graph of size q and let f be a labeling of G. If the set of the labels of G is a subset of $\{0, 1, 2, \dots, q\}$ and the set of the weights is $\{1, 2, \dots, q\}$ then, the labeling f is said to be *graceful* and G is called a *graceful* graph. Note that each graceful labeling of a graph G is a complete 1–equitable labeling of G. Graceful labelings of graphs were introduced independently by A. Rosa [12] and S.W. Golomb [8]. Surveys on graceful labelings can be found in [2], [4] and [6].

The existence of an optimal complete k–equitable labeling of a graph G is also related to the so–called Skolem–graceful graphs. A graph G of order p and size q is said to be *Skolem–graceful* if there exists a labeling f of G such that $f(V(G)) = \{1, 2, \dots, p\}$ and the set of weights induced by f on the edges of G is $\{1, 2, \dots, q\}$. Thus, a graph G is Skolem–graceful if and only if there exists an optimal complete 1–equitable labeling of G. Skolem–graceful graphs were introduced by S. M. Lee and S. C. Shee in [9].

2. Preliminary results

In [3], Bloom and Ruiz proved that every graph G is 1–equitable by assigning distinct powers of 2 to the vertices of G. They also observed that

The last theorem may be used to demonstrate that a graph G is not k–equitable, for some values of k.

Corollary 2. *Suppose that G is an r–regular graph of order p which is also k–equitable. Then, $k \leq \frac{p}{2}$.*

For example, for $k \geq 3$, let G_k be the graph on $2(k-1)$ vertices such that $V(G_k) = \{v_1, v_2, \dots, v_{2(k-1)}\}$. The edge–set of G_k contains the edges of the cycle $v_1, v_2, \dots, v_{2(k-1)}, v_1$ and the edges of the two complete graphs whose vertex–sets are $\{v_1, v_3, \dots, v_{k-1}\}$ and $\{v_2, v_4, \dots, v_{2(k-1)}\}$, respectively. So, the size of G_k is $k(k-1)$. An application of Corollary 2 implies that G_k is not k–equitable. In Figure 2, we show the graph G_k, for $k = 4$ and $k = 5$.

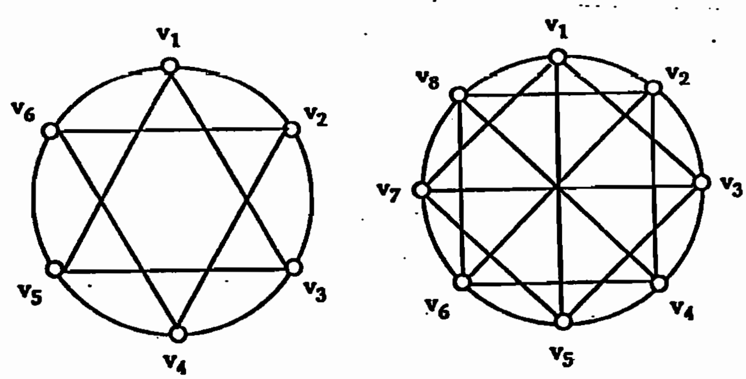


Figure 2

We know of graphs that are not k–equitable though, they satisfy both degree conditions given in Theorem B and Theorem 1. For example, the bipartite graphs $K(2,3)$ and $K(3,3)$ are not 3–equitable; the graph $K(2,4)$ is not 4–equitable. Another example is the graph $K_6 - e$ which is not 2–equitable. Another example is the graph $k_6 - e$ which is not 2–equitable, as we mentioned earlier. For $k = 5$, it is known (see [1]) that the tree shown in Figure 3 is not 5–equitable.

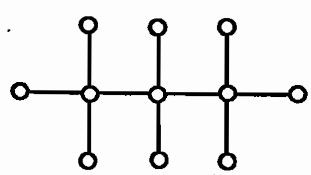


Figure 3

the complete graph K_p, $p \geq 2$, is k–equitable if and only if $k = 1$. From this observation, it is clear that not every graph of even size is 2–equitable. Consider, for example, a nontrivial complete graph of even size. Actually, K_4 is the smallest graph in order and size that is not 2–equitable. If we restrict our attention to noncomplete graphs then, we know that the graph $K_6 - e$ (e and edge of K_6) is the unique graph with the smallest order that is not 2–equitable. That is, if G is a noncomplete graph of order smaller than 7 and G is not isomorphic to $K_6 - e$ then, G must be 2–equitable. On the other hand, in [1], the following theorem was proved.

Theorem A. *Every forest of even size is 2–equitable.*

In [1], it was also proved that every caterpillar of even size has an optimal complete 2–equitable labeling. Indeed, we know that if T is a tree of even size $q \leq 10$ then, T has an optimal complete 2–equitable labeling. We believe that the following conjecture is true.

Conjecture . Every tree of even size has an optimal complete 2–equitable labeling.

Before going to the next section where we present new families of 2–equitable graphs, we state two necessary conditions for the existence of k–equitable labelings of graphs.

In [1], it was established that if a graph G is k–equitable then the maximum degree $\Delta(G)$ of G is bounded from above. We state this condition in the following theorem.

Theorem B. *If G is a k–equitable graph of size $q = kw$ then, $\Delta(G) \leq 2w$.*

If $k = 3$ and $k = 4$, the degree constraint given in Theorem B is a sufficient condition for the k–equitability of a forest whose size is divisible by k, as it was shown in [1].

Theorem C. *For $k = 3$ and $k = 4$, let F be a forest of size $q = kw$. Then, the forest F is k–equitable if and only if $\Delta(G) \leq 2w$.*

Another necessary condition for the existence of a k–equitable labeling of a graph G involves the minimum degree $\delta(G)$ of G. This result is presented next.

Theorem 1. *If G is a k–equitable graph of size $q = kw$ then, $\delta(G) \leq w$.*

Proof. Let f be a k–equitable labeling of G and assume that v is that vertex of G that has the minimum label. It follows that v cannot be incident with two edges of G that have he same weight h. (Otherwise, there exists a vertex $u \in V(G)$ such that $f(u) = f(v) - h$, which is not possible.) Therefore, deg v is at most equal to w, where w is the number of distinct weights induced by f on the edges of G. □

Observe that the proof of Theorem 1 implies that every k–equitable graph of size $q = kw$ contains at least two vertices of degree at most equal to w.

3. On the 2-equitability of some families of graphs

In this section we discuss the 2-equitability of certain families of graphs. We begin showing that the graphs called prisms are 2-equitable.

The *prism* D_n, $n \geq 3$, is the Cartesian product $C_n \times K_2$ of the cycle C_n and the complete graph of order 2. Since the size of D_n is $3n$, if D_n is 2-equitable then n must be even.

In order to prove that the prism D_n is 2-equitable when n is even, we need two labelings of the cycle C_n.

In 1967, A. Rosa [12] proved that the cycle C_n is graceful if and only if $n \equiv 0$ or $n \equiv 3 \pmod 4$. When $n \equiv 1$ or $n \equiv 2 \pmod 4$, S.P. Rao Hebbare (see [11]) found a proper labeling of C_n for which the greatest label assigned to the vertices of G is $n+1$ and such that, the set of weights is $\{2,3,4,\ldots,n+1\}$, if $n \equiv 1 \pmod 4$, and the set $\{1,3,4,\ldots,n+1\}$, if $n \equiv 2 \pmod 4$. Next, we present the graceful labeling of C_n found by Rosa and the 1-equitable labeling of C_n found by Rao Hebbare.

Assume that $v_1, v_2, \ldots, v_n, v_n, v_1$ are the consecutive vertices of the cycle C_n. For $n \equiv 0 \pmod 4$, the following labeling f of C_n is graceful.

$$f(v_i) = \begin{cases} \dfrac{i-1}{2} & i \text{ odd} \\[2ex] n+1-\dfrac{i}{2} & i \text{ even}, i \leq \dfrac{n}{2} \\[2ex] n-\dfrac{i}{2} & i \text{ even}, i > \dfrac{n}{2} \end{cases} \tag{3.1}$$

When $n \equiv 2 \pmod 4$, let f be the following labeling of C_n.

$$f(v_i) = \begin{cases} n+1-\dfrac{i}{2} & i \text{ even}, i \leq n-2 \\[2ex] n+1 & i \text{ even}, i = 1 \\[2ex] \dfrac{i-1}{2} & i \text{ odd}, i \leq \dfrac{n}{2} \\[2ex] \dfrac{i+1}{2} & i \text{ odd}, \dfrac{n}{2} < i < n-1 \\[2ex] \dfrac{n+2}{2} & i \text{ odd}, i = n-1 \end{cases} \tag{3.2}$$

Theorem 3. *If n is even, the prism D_n is 2-equitable.*

Proof. Let $U : u_1, u_2, \ldots, u_n, u_1$ and $V : v_1, v_2, \ldots, v_n, v_1$ be two vertex-disjoint cycles of D_n such that u_i is adjacent to v_i, for all $i = 1, 2, \ldots, n$.

Let f be the labeling of C_n described in (1) or (2), according to n. Let g be the following labeling of D_n.

$$\begin{aligned} g(v_i) &= f(v_i), \\ g(u_i) &= k - f(v_{n-i+1}), \end{aligned}$$

for $i = 1, 2, \dots, n$, where

$$k = \begin{cases} 10t & \text{if } n = 4t \\ 10t + 8 & \text{if } n = 4t + 2 \end{cases}$$

It is straightforward to see that g induces on the edges of U the same weights that g induces on the edges of V. On the other hand, for each $i = 1, 2, \dots, n$, let w_i be the weight of the edge $u_i v_i$. That is, $w_i = |g(u_i) - g(v_i)|$. Then,

$$w_i = w_{n-i+1}, \quad \text{for } i = 1, 2, \dots, \frac{n}{2}.$$

It remains to prove that the weights $w_1, w_2, \dots, w_{\frac{n}{2}}$ are all different and that none of them appears as a weight of an edge of $U \cup V$. We divide the rest of the proof into two cases.

Case 1: $n = 4t$, for some integer t.

In this case,

$$w_i = g(u_i) - g(v_i) = \begin{cases} 4t - 1 + i, & i \le \frac{n}{2}, i \text{ even} \\ \\ 8t + 1 - 2, & i < \frac{n}{2}, i \text{ odd} \end{cases}$$

Therefore, the weights w_i, $i = 1, 2, \dots, \frac{n}{2}$, are all different and the smallest of them is $4t + 1 = n + 1$. Since n is the largest weight induced by g on the edges of $U \cup V$, it follows that g is 2–equitable and so is D_n.

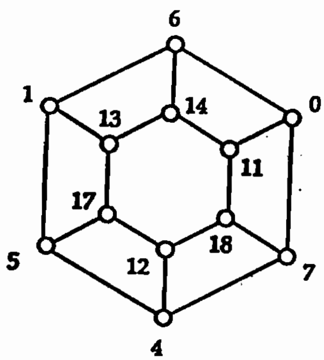
Case 2: $n = 4t + 2$, for some integer t.

In this second case,

$$w_i = g(u_i) - g(v_i) = \begin{cases} 4t + 4, & i = 2 \\ \\ 4t + 3 + i, & 4 \le i < \frac{n}{2}, i \text{ even} \\ \\ 8t + 7 - i, & 3 \le i \le \frac{n}{2}, i \text{ odd} \\ \\ 6t + 5, & i = 1 \end{cases}$$

Again, the weights w_i, $i = 1, 2, \dots, \frac{n}{2}$, are all different being the smallest of them $4t + 4 = n + 2$. As in the previous case, we obtain that D_n is 2–equitable. □

In Figure 4 we show 2–equitable labelings for the prisms D_6 and D_8.



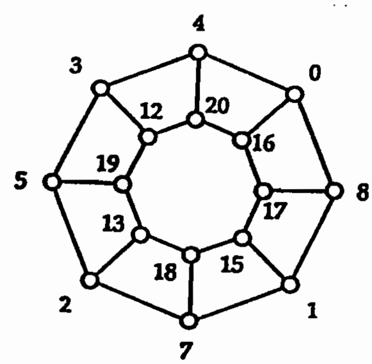


Figure 4

In 1979, M. Maheo [10] proved that the n–cube Q_n is graceful, answering a conjecture of J.C. Bermond [2] and of T. Gangopadhyay and Rao Hebbare [7]. We will use this result to prove that there exists a complete 2–equitable labeling of the cube Q_n.

The *cube* Q_n can be described as the graph whose vertex–set s the set of the binary n–tuples $(b_1b_2\dots b_n)$ and such that two vertices are adjacent if and only if their corresponding n–tuples differ in precisely one position. Let f_n be the graceful labeling of Q_n given by Maheo in [10]. (See an example of this graceful labeling in Figure 5.)

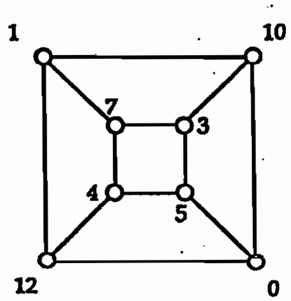


Figure 5

Next, we use f_{n-1} to define a 2–equitable labeling g_n of Q_n.

$$g_n(b_1b_2\dots b_{n-1}0) = f_{n-1}(b_1b_2\dots b_{n-1}) + 1$$
$$g_n(b_1b_2\dots b_{n-1}1) = (n-1)\cdot 2^{n-1} + 2 - f((1-b_1)b_2\dots b_{n-1})$$

Observe that the correspondence

$$(b_1b_2\dots b_n) \to ((1-b_1)b_2\dots b_n)$$

defines an automorphism of Q_{n-1}. Let $U = \{(b_1b_2\dots b_n)|b_n = 0\}$ and $V = \{(b_1b_2\dots b_n)|b_n = 1\}$. Note that g_n induces on the edges of $\langle U\rangle$ the same weights that g_n induces on the edges of $\langle V\rangle$. Namely, the weights $1, 2, \dots, (n-1)\cdot 2^{n-2}$. Let $w(b_1b_2\dots b_{n-1})$ be the weight of the edge $(b_1b_2\dots b_{n-1}0)(b_1b_2\dots b_{n-1}1)$ of Q_n. Then

$$w(b_1b_2\dots b_{n-1}) = |g_n(b_1b_2\dots b_{n-1}0) - g_n(b_1b_2\dots b_{n-1}1)|,$$

and we easily obtain that $w(b_1b_2\dots b_{n-1}) = w((1-b_1)b_2\dots b_{n-1})$. Using induction and the properties of the graceful labeling f_n of Q_n, it is possible to prove that the weights $w(b_1b_2\dots b_{n-1})$ form the set $\{(n-1)\cdot 2^{n-2}+1, (n-1)\cdot 2^{n-2}+2, \dots, n\cdot 2^{n-2}\}$. Therefore, g_n is a complete 2–equitable labeling of Q_n. We state this result in our next theorem.

Theorem 4. *For each positive integer $n \geq 2$, the n–cube Q_n has a complete 2–equitable labeling.*

An example of a complete 2–equitable labeling of Q_4 is shown in Figure 6.

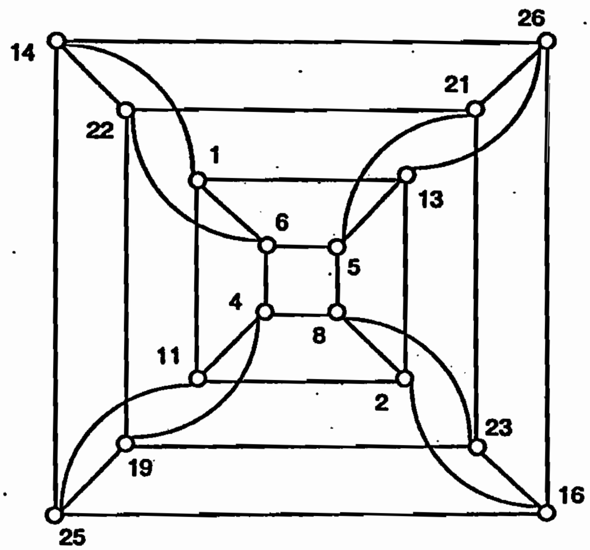


Figure 6

For $n \geq 3$, the *book* B_n is the Cartesian product $S_n \times K_2$, where S_n is the star with n end–vertices and K_2 is the complete graph on two vertices. The graph B_n consists of n cycles of length 4 that share a common edge. The size of B_n is $3n+1$ so, B_n has an even number of edges when n is odd. In the next theorem we will prove that if n is odd, B_n is 2–equitable.

Theorem 5. *If $n \geq 3$ is an odd integer, the book B_n has a 2–equitable labeling.*

Proof. For $0 \le i \le n$ and $j = 0, 1$, denote the $2(n+1)$ vertices of B_n by v_i^j, where, for each $j = 0, 1$, v_0^j is adjacent to v_i^j for every $i = 1, 2, \dots, n$ and, v_i^0 is adjacent to v_i^1 for all $i = 0, 1, \dots, n$. Let f be the proper labeling of B_n defined as follows:

$$f(v_i^j) = \begin{cases} 4n-10 & i=0, & j=0 \\ 8i-7 & i \le i \le \frac{n-1}{2}, & j=0 \\ 4n-3 & i = \frac{n+1}{2}, & j=0 \\ 4(4i-n-7), & \frac{n+3}{2} \le i \le n, & j=0 \\ 4n-7, & i=0, & j=1 \\ 8(n-i)-9 & 1 \le i \le \frac{n-1}{2}, & j=1 \\ 4n, & i=\frac{n+1}{2}, & j=1 \\ 8i-18, & \frac{n+3}{2} \le i \le n, & j=1 \end{cases}$$

The weights obtained on the edges $v_0^0 v_i^0$, for $i = 1, 2, \dots, n$, form the set $\{1, 9, 17, \dots, 4n-11, 7, 6, 22, 38, \dots, 8n-18\}$;
the weights obtained on the edges $v_0^1 v_i^1$, for $i = 1, 2, \dots, n$, form the set $\{1, 9, 17, \dots, 4n-11, 7, 2, 10, 18, \dots, 4n-10\}$; the weights obtained on the edges $v_i^0 v_i^1$, for $i = 1, 2, \dots, n$, form the set
$\{2, 10, 18, \dots, 4n-10, 3, 6, 22, 38, \dots, 8n-18\}$;
and the weight obtained on the edge $v_0^0 v_0^1$ is 3. Since each weight induced by f appears twice on the edges of B_n, the labeling f is 2–equitable. □

In Figure 7, we show a 2–equitable labeling of the book B_7.

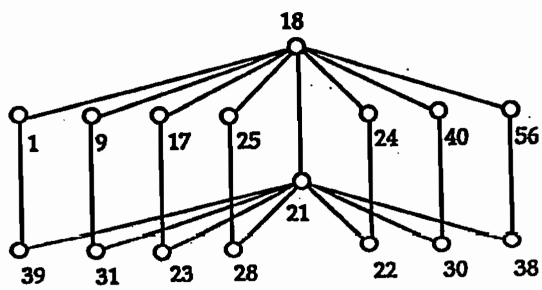


Figure 7

Let $v_1, v_2, \dots, v_n$ be the consecutive vertices of a cycle C_n. The *wheel* W_n is the graph obtained from C_n by adding a new vertex v_0 and the edges $v_0 v_i$, for $i = 1, 2, \dots, n$. (These new edges are called the *spokes* of W_n.) Our next theorem presents partial results about the 2–equitability of wheels.

Theorem 6. *The wheel W_n is 2–equitable when n is even or $n \equiv 1 \pmod 4$.*

Proof. We use the notation previously given to define a 2–equitable labeling f of W_n. We consider two cases.

Case 1. $n = 2t$ where $t \geq 2$.

$$\begin{aligned} f(v_0) &= 2t & \\ f(v_{2i-1}) &= i-1 & 1 \leq i \leq [\tfrac{t}{2}] \\ f(v_{2i}) &= t-i & 1 \leq i \leq [\tfrac{t}{2}] \\ f(v_{t+2i-1}) &= 4t-i+1 & 1 \leq i \leq [\tfrac{t}{2}] \\ f(v_{t+2i}) &= 3t+i & 1 \leq i \leq [\tfrac{t}{2}] \end{aligned}$$

In this case, on the spokes of the wheel, we get the weights $t+1, t+2, \ldots, 2t$, each of them repeated twice. On the edges of the external cycle we get the weights $1, 2, \ldots, t-1$ and the weight $\frac{7t}{2}$ if t is even or, $\frac{7t+1}{2}$ if t is odd. Similarly, each weight is repeated twice. (In Figure 8, we show an example of this labeling for $n = 12$.)

Case 2: $n \equiv 1 \pmod 4$, i.e. $n = 2t+1$ where $t \equiv 0 \pmod 2$, $t \geq 2$.

$$\begin{aligned} f(v_0) &= 2t & \\ f(v_{2i-1}) &= i-1 & 1 \leq i \leq [\tfrac{t+2}{2}] \\ f(v_{2i}) &= t+1-i & 1 \leq i \leq [\tfrac{t}{2}] \\ f(v_{t+2i}) &= 4t-i+1 & 1 \leq i \leq [\tfrac{t}{2}] \\ f(v_{t+2i+1}) &= 3t+i & 1 \leq i \leq [\tfrac{t}{2}] \end{aligned}$$

This time, on the spokes of W_n we obtain the weights $t+1, t+2, \ldots, 2t$, each repeated twice and the weight t appearing exactly once. On the edges of the external cycle, we get the weights $1, 2, \ldots, t-1$, each repeated twice and the weight t appearing exactly once. (In Figure 8 we show an example of this labeling for $n = 13$.) □

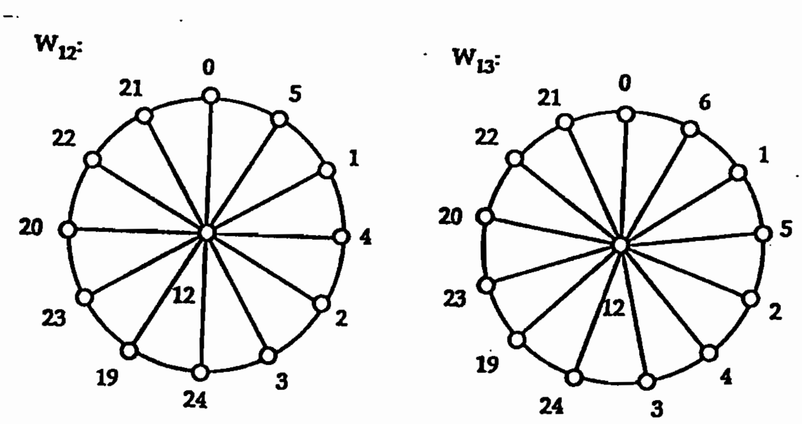


Figure 8

Recall that $W_3 = K_4$ is not 2–equitable; however, we have enough evidence to state the following conjecture.

Conjecture . The wheel W_n is 2–equitable when $n \equiv 3 \pmod 4$, $n \geq 7$

In Figure 9 we show 2–equitable labelings of the wheels W_7 and W_{11}.

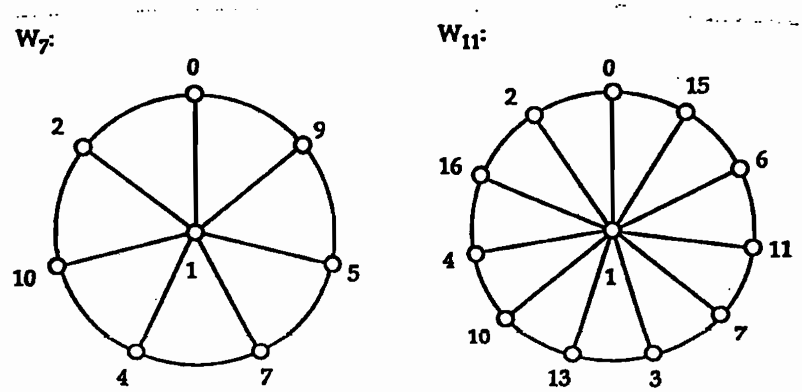


Figure 9

To finish this section, we prove that complete bipartite graphs of even size accept 2–equitable labelings that are complete.

Theorem 7. *Let m and n be positive integers. If m or n are even then, $K(m,n)$ has a complete 2–equitable labeling.*

Proof. Let M and N be the partite sets of $G = K(m,n)$, where $|M| = m$ and $|N| = n$. Without loss of generality, we may assume that m is even. Let M_1 and M_2 be two disjoint subsets of M such that $|M_1| = |M_2| = \frac{m}{2}$. Assign the integers $0, 1, \ldots, \frac{m}{2} - 1$ to the vertices of the set M_1, the integers $\frac{mn}{2} + 1, \frac{mn}{2} + 2, \ldots, \frac{mn}{2} + \frac{m}{2}$ to the vertices of the set M_2, and the integers $\frac{m}{2}, m, \frac{3m}{2}, \ldots, \frac{mn}{2}$ to the vertices of N. It is straightforward to check that the weights induced by this labeling of G are the integers $1, 2, \ldots, \frac{mn}{2}$, and that each of these weights appears exactly twice on the edges of G. □

Two examples of the labeling utilized in the proof of Theorem 7 are shown in Figure 10.

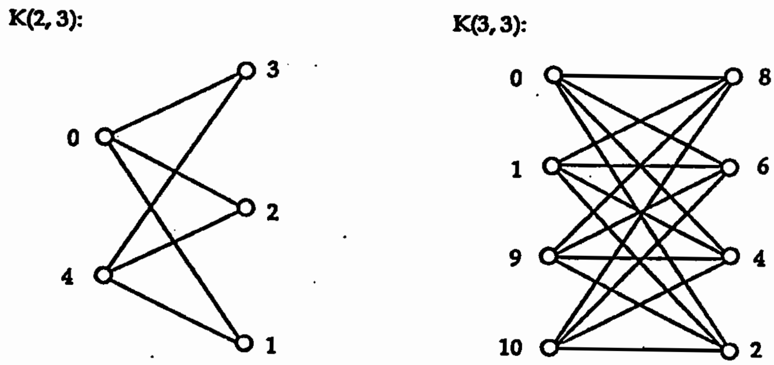


Figure 10

4. Constructing k–equitable graphs

First, we briefly remark two ways to construct k–equitable graphs. In what follows, we assume that the graphs G and H are vertex–disjoint.

Suppose that G and H are k–equitable graphs and let g and h be k–equitable labelings of G and H, respectively. Further, suppose that g and h induce the same weights on the edges of the corresponding graphs. Then, the graph $G \cup H$ is $(2k)$–equitable. Let u be the vertex of G that has the greatest label and let v be the vertex of H that has the smallest label. Then, the graph $(G, u = v, H)$ obtained from G and H by identifying the vertices u and v is $(2k)$–equitable. See examples of these constructions in Figure 11.

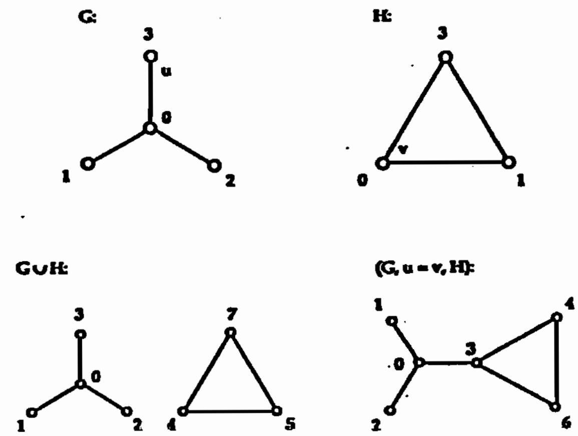


Figure 11

Under no other restrictions, if G and H are two vertex–disjoint k–equitable graphs then, the graph $G \cup H$ is k–equitable. (See an example in Figure 12). We state this result in a more general form.

Theorem 8. *Let $G_1, G_2, \ldots, G_n$ be n vertex–disjoint k–equitable graphs. Then the graph $G_1 \cup G_2 \cup \ldots \cup G_n$ is k–equitable.*

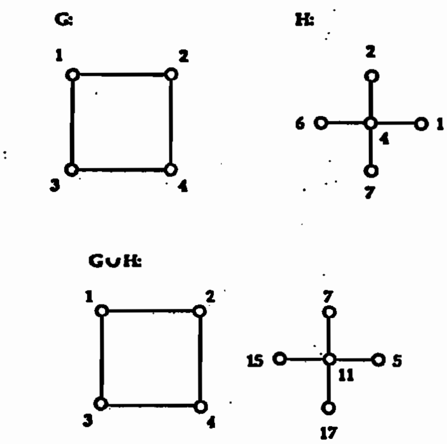


Figure 12

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