

# Equitable Labelings of Forests

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*In memory of Sergio Ruiz*

## ABSTRACT

A *difference labeling* of a graph  $G$  is an assignment of distinct integers to the vertices of  $G$ . If  $f$  is a difference labeling of  $G$ , the *weight* of an edge  $uv \in E(G)$  is the integer  $|f(u) - f(v)|$ . A labeling  $f$  of a graph  $G$  is said to be *k-equitable* if for each weight  $h$  induced by  $f$  there are exactly  $k$  edges of  $G$  that have weight  $h$ . Equitable labelings of graphs are generalizations of the well-known graceful labelings of graphs and, they were introduced recently by Bloom and Ruiz. A graph  $G$  of size  $q$  is said to be *equitable* if  $G$  is  $k$ -equitable for each divisor  $k$  of  $q$ ,  $k \neq q$ . It is known that cycles are equitable. We show that linear forests are another family of equitable graphs. We prove that forests of even size are 2-equitable. Let  $w$  be a positive integer. For  $k = 3$  and  $k = 4$ , we show that if  $F$  is a forest of size  $kw$  and maximum degree  $\Delta(F)$  then,  $F$  is  $k$ -equitable if and only if  $\Delta(F) \leq 2w$ . We discuss the  $k$ -equitability of forests for  $k \geq 5$  and characterize all those caterpillars of diameter 2 that are  $k$ -equitable, for all possible value of  $k$ .

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## 1. Introduction and basic results

The terminology and notation are taken from [5]. In general, graphs considered here have no isolated vertices. A *difference labeling* of a graph  $G$  is an assignment of distinct integers to the vertices of  $G$ . In this paper the term *labeling* is used to abbreviate "difference labeling". Assume that  $f$  is a labeling of  $G$ . The *weight* of an edge  $uv \in E(G)$  is the integer  $|f(u) - f(v)|$ . A labeling of a graph  $G$  is called *proper* if all the labels of the vertices of  $G$  are non-negative. Observe that every labeling  $f$  of a graph  $G$  can be transformed into a proper labeling of  $G$  that preserves the weights induced by  $f$ . Thus, properties that hold for labelings of graphs can be also stated for proper labelings of graphs.

A labeling  $f$  of a graph  $G$  is *k-equitable* if for each weight  $h$  induced by  $f$  there are exactly  $k$  edges of  $G$  that have weight  $h$ . In this case, the graph  $G$  is said to be *k-equitable*. For example, in Figure 1 we show a 2-equitable labeling of  $K_{2,3}$ . Equitable labelings of graphs were introduced by Bloom and Ruiz in [3].

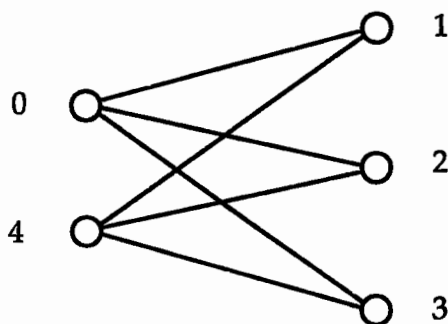


Figure 1

Suppose that  $f$  is a  $k$ -equitable labeling of a graph  $G$  and let  $h$  denote the weight of an edge of  $G$ . Let  $S_h$  be the subgraph of  $G$  induced by the edges of  $G$  that have weight  $h$  and assume that  $u$  and  $v$  are two adjacent vertices of  $S_h$ . Since

$$|f(u) - f(v)| = h \Leftrightarrow f(u) = f(v) + h \text{ or } f(u) = f(v) - h,$$

$\deg_{S_h} u \leq 2$  and hence,  $\Delta(S_h) \leq 2$ . Observe that each component  $C$  of  $S_h$  can be embedded in the real line  $L$  by corresponding every vertex  $u \in V(C)$  with that point of  $L$  whose coordinate is  $f(u)$ . Therefore,  $C$  is a path and  $S_h$  is a linear forest. Let  $W$  denote the set of the weights induced by  $f$ . Then, it follows from the previous observation that  $\{S_h\}_{h \in W}$  is a decomposition of  $G$  into linear

forests of size  $k$ . With the help of this observation, we establish the next two results.

**Lemma 1** Let  $G$  be a graph of size  $q > 0$ . Then,  $G$  is  $q$ -equitable if and only if  $G$  is a linear forest.

**Proof**  $\Rightarrow$ ) Since  $G$  is  $q$ -equitable, there exists an equitable labeling of  $G$  in which all the edges of  $G$  have the same weight  $h$ . Therefore  $G \cong S_h$ , which is a linear forest.

$\Leftarrow$ ) Use consecutive integers to label the vertices of each component of the linear forest  $G$ .  $\square$

Let  $G$  be a graph of size  $q$  and let  $f$  be a  $k$ -equitable labeling of  $G$  that induces exactly  $w$  distinct weights on the edges of  $G$ . Then  $q = kw$  and, as we state next, the maximum degree of  $G$  is bounded above by  $2w$ .

**Lemma 2 (of the degree constraint)** Let  $G$  be a  $k$ -equitable graph of size  $q$ . Then

$$\Delta(G) \leq 2w,$$

where  $w = \frac{q}{k}$ .

**Proof** Let  $f$  be a  $k$ -equitable labeling of  $G$  and assume that  $W$  is the set of the weights induced by  $f$  on the edges of  $G$ . Then,  $|W| = w = \frac{q}{k}$ . For  $h \in W$ , let  $S_h$  denote the subgraph of  $G$  induced by the edges of  $G$  that have weight  $h$ . Let  $v \in V(G)$ . Since the collection  $\{S_h\}_{h \in W}$  of subgraphs of  $G$  is a decomposition of  $G$  into linear forests,  $\deg_{S_h} v \leq 2$  and hence,  $\deg_G v \leq 2w$ , for each vertex  $v \in V(G)$ .  $\square$

Let  $G$  be a graph of size  $q$ . Then  $G$  is said to be *equitable* if  $G$  is  $k$ -equitable for each divisor  $k$  of  $q$ ,  $k \neq q$ . In [11], J. Wojciechowski proved that cycles are equitable. (Bloom and Ruiz had mentioned this property in [3], and C. Barrientos gave a proof of this fact in [1].) In the following section we present another family of equitable graphs.

## 2. Linear forests are equitable

Suppose that  $G$  is a graph of size  $q$  and let  $f$  be a labeling of  $G$ . If the set of the labels of  $G$  is a subset of  $\{0, 1, 2, \dots, q\}$  and the set of the weights is  $\{1, 2, \dots, q\}$  then, the labeling  $f$  is called *graceful* and  $G$  is said to be a *graceful graph*. Graceful labelings of graphs have been studied extensively since they were introduced by Rosa [9] and Golomb [7]. To this respect, it is still unknown whether all trees are graceful. (See the related surveys [2], [4], and [6].)

Let  $G$  be a graph of size  $q$  and suppose that  $f$  is a  $k$ -equitable labeling of  $G$  that induces exactly  $w$  distinct weights. If the set of the  $w$  different weights induced by  $f$  is  $\{1, 2, \dots, w\}$  then,  $f$  is said to be *complete*. Thus, each graceful labeling of a graph  $G$  is a complete 1-equitable labeling of  $G$ . In Figure 2 we give a complete 1-equitable labeling of the *nongraceful* graph  $K_2 \cup K_3$ . This shows that not every complete 1-equitable labeling of a graph  $G$  is a graceful labeling.

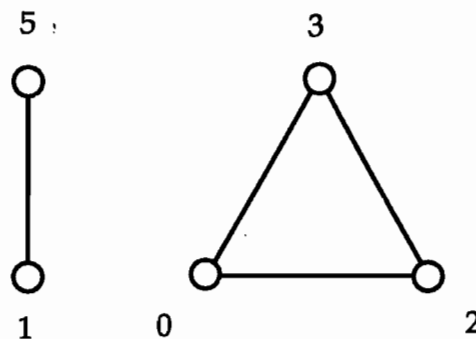


Figure 2

A labeling of a graph  $G$  is said to be *optimal* if the labels of the vertices of  $G$  are consecutive integers. (Note that the equitable labeling of  $K_{2,3}$  shown in Figure 1 is optimal.) Thus, if  $T$  is a tree then,  $T$  is graceful if and only if there exists an optimal complete 1-equitable labeling of  $T$ .

It is well-known that paths are graceful graphs. (See [9].) For example, if  $P: v_0, v_1, \dots, v_n$  is a path of size  $n$  then, the labelings  $\gamma_n^0$  and  $\gamma_n^1$  defined below are two graceful labelings of  $P$ . These labelings are called the *canonical graceful labelings* of the paths.

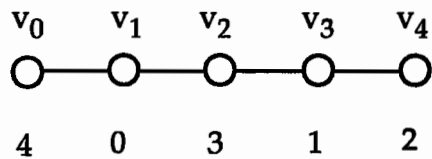
$$\gamma_n^0(v_{2i+1}) = i \quad 0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$$

$$\gamma_n^0(v_{2i}) = n - i \quad 0 \leq i \leq \lceil \frac{n-1}{2} \rceil$$

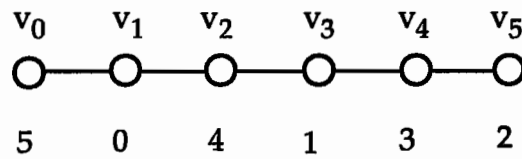
$$\gamma_n^1(v_j) = n - \gamma_n^0(v_j) \quad 0 \leq j \leq n.$$

In Figure 3, we show the labelings  $\gamma_n^0$  and  $\gamma_n^1$  for  $n = 4$  and  $n = 5$ .

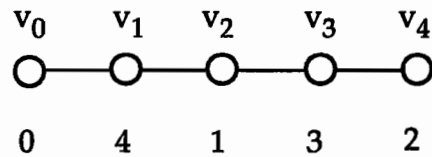
$\gamma_4^0$ :



$\gamma_5^0$ :



$\gamma_4^1$ :



$\gamma_5^1$ :

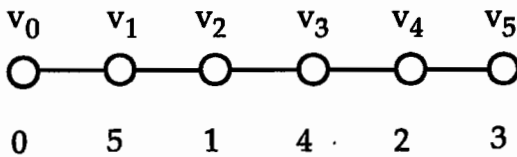


Figure 3

The existence of the graceful labeling  $\gamma_n^0$  implies the existence of an optimal complete 1-equitable labeling of the path  $P_{n+1}$ . A generalization of this result is now presented.

**Theorem 3** For each positive integer  $n \geq 1$  and for each divisor  $k$  of  $n$  there exists an optimal complete  $k$ -equitable labeling of the path  $P_{n+1}$ .

**Proof** Since the result is clear for  $k = 1$  and  $k = n$ , we may assume  $1 < k < n$ .

Let  $w = \frac{n}{k}$ . We use a recursive definition to describe a complete  $k$ -equitable labeling  $f$  of the path  $P: v_0, v_1, \dots, v_n$ .

Case 1:  $w \equiv 0 \pmod{2}$ .

$$f(v_i) = \gamma_w^0(v_i) \quad \text{if } 0 \leq i \leq w,$$

$$f(v_i) = 2w - f(v_{2w+1-i}) \quad \text{if } w+1 \leq i \leq 2w,$$

and, when  $k > 2$ ,

$$f(v_i) = w + f(v_{2tw+1-i}) \quad \text{if } tw+1 \leq i \leq (t+1)w, \\ \text{where } 2 \leq t \leq k-1.$$

Case 2:  $w \equiv 1 \pmod{2}$ .

$$f(v_i) = \gamma_w^1(v_i) \quad \text{if } 0 \leq i \leq w,$$

$$f(v_i) = 2w+1 - f(v_{2w+1-i}) \quad \text{if } w+1 \leq i \leq 2w,$$

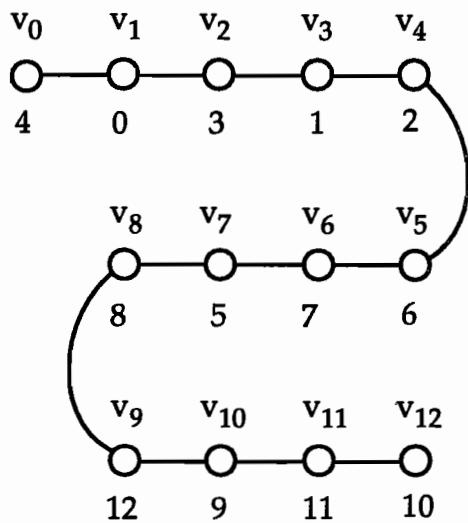
and, when  $k > 2$ ,

$$f(v_i) = w + f(v_{2tw+1-i}) \quad \text{if } tw+1 \leq i \leq (t+1)w, \\ \text{where } 2 \leq t \leq k-1.$$

□

In Figure 4, and for  $k = 3$  and  $k = 4$ , we show a  $k$ -equitable labeling of a path  $P$  of size 12. These labelings are examples of the labeling  $f$  defined in the proof of Theorem 3. Note that in each of the graphs of Figure 4, the vertices  $v_0, v_1, \dots, v_w$  of  $P$  have been placed forming a row. The remaining vertices of  $P$  are drawn below forming  $k-1$  rows, each containing the vertices  $v_{tw+2}, v_{tw+3}, \dots, v_{(t+1)w+1}$ ,  $1 \leq t \leq k-1$ , respectively. In these drawings of  $P$  we may also observe that the  $k$ -labeling  $f$  repeats, level by level, the pattern of labels that the canonical graceful labelings  $\gamma_n^0$  and  $\gamma_n^1$  present.

$k = 3$   
 $w = 4$



$k = 4$   
 $w = 3$

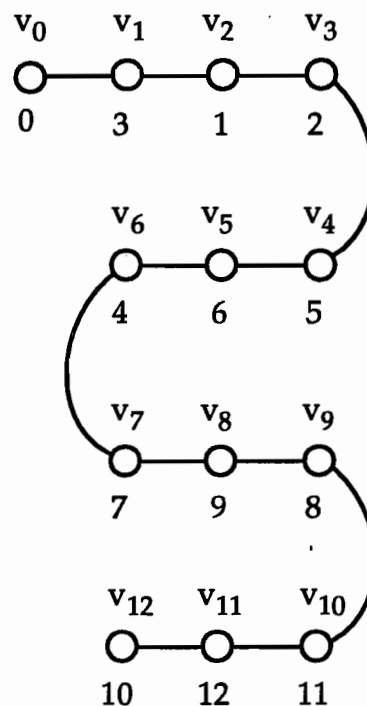


Figure 4

From Theorem 3, it follows that paths are optimal equitable graphs. This result can be partially extended to linear forests, as we show next.

**Corollary 4** Let  $F$  be a linear forest of size  $q$  and suppose that  $k$  divides  $q$ . Then, there exists a complete  $k$ -equitable labeling of  $F$ .

**Proof** Suppose that  $F = \bigcup_{i=1}^m P_{q_i+1}$  is a linear forest of size  $q$  which is

isomorphic to the (disjoint) union of  $m$  paths  $P_{q_i+1} : v_0^{(i)}, v_1^{(i)}, \dots, v_{q_i}^{(i)}$ ,

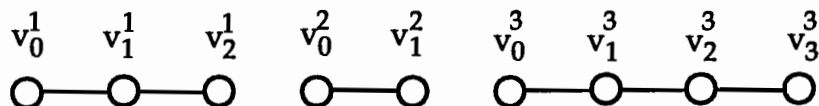
$1 \leq i \leq m$ . (So,  $q = \sum_{i=1}^m q_i$ .) We may assume that  $q_i > 0$ , for each  $i = 1, 2, \dots, t$ . Let

$k$  be a divisor of  $q$ . First, we find a proper complete  $k$ -equitable labeling  $f$  of the path  $P : v_0, v_1, \dots, v_q$ . Let  $M = \max \{f(v_i) \mid i = 0, 1, \dots, q\}$ . We use  $f$  to define a complete  $k$ -equitable labeling  $g$  of  $F$ .

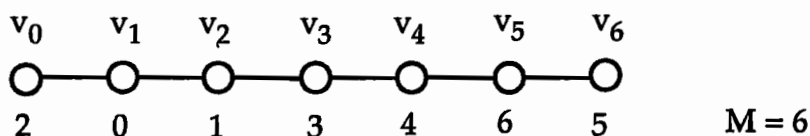
$$g(v_j^i) = \begin{cases} f(v_j) & 0 \leq j \leq q_1 \text{ and } i = 1, \\ f(v_{q_1 + q_2 + \dots + q_{i-1} + j}) + (i-1)(M+1) & 0 \leq j \leq q_i \text{ and } 2 \leq i \leq m. \end{cases} \quad \square$$

In Figure 5, we use the construction given in the proof of Corollary 4 to find a complete 3-equitable labeling  $g$  of a forest  $F \cong P_3 \cup P_2 \cup P_4$ .

F:



$P_6$ :



$g$ :

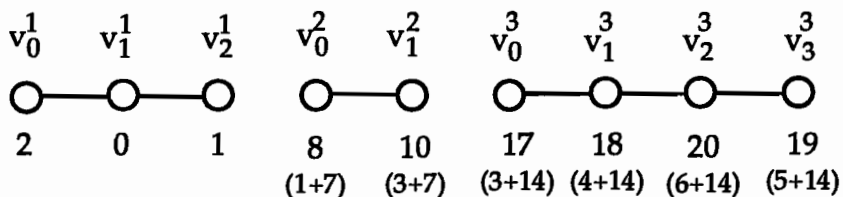


Figure 5

The existence of an optimal complete  $k$ -equitable labeling of a graph  $G$  is also related to the so-called Skolem-graceful graphs introduced by Lee and Shee in [8].

A graph  $G$  of order  $p$  and size  $q$  is said to be *Skolem-graceful* if there exists a labeling  $f$  of  $G$  such that  $f(V(G)) = \{1, 2, \dots, p\}$  and the set of weights induced by  $f$  on the edges of  $G$  is  $\{1, 2, \dots, q\}$ .

Note that a graph  $G$  is a Skolem-graceful graph if and only if there exists an optimal complete 1-equitable labeling of  $G$ . Using a result from Skolem [10], it follows that the graph  $nP_2$  is Skolem graceful if and only if  $n \equiv 0$  or

$n \equiv 0 \pmod{4}$ . We now state the existence of an optimal  $k$ -equitable labeling of the graph  $nP_2$ , for each divisor  $k$  of  $n$ .

**Theorem 5** Let  $G \cong nP_2$  and suppose that  $k$  is a divisor of the size  $n$  of  $G$ . Then, there exists an optimal  $k$ -equitable labeling of  $G$ .

In Figure 6 we show an optimal 3-equitable labeling of the graph  $12P_2$ . This example may also suggest a procedure to obtain such a labeling, in the general case stated in Theorem 5.

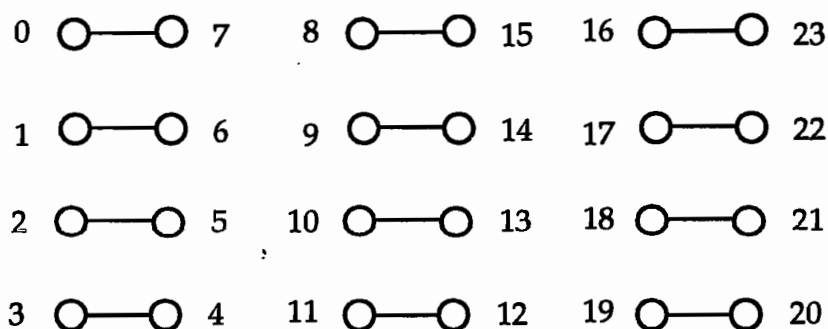


Figure 6

### 3. The $k$ -equitability of forests, for $k = 2, 3$ , and $4$ .

In [3], Bloom and Ruiz proved that all graphs are 1-equitable. They also proved that the complete graph  $K_p$ ,  $p \geq 2$ , is not  $k$ -equitable for each  $k > 1$ . We may obtain this last result, observing that if  $f$  is a  $k$ -equitable labeling of a complete graph  $G$  then, the weight of the edge joining the vertex with the maximum label and the vertex with the minimum label is unique, which implies that  $k = 1$ .

The smallest graph of even size that is not 2-equitable is  $K_4$ . In this section we prove that each forest of even size is 2-equitable. Moreover, for  $k = 3$  and for  $k = 4$ , we give necessary and sufficient conditions for the  $k$ -equitability of a forest whose size is divisible by  $k$ .

Let  $F$  be a forest. A subset  $R \neq \emptyset$  of edges of  $F$  is called a *linear connector* of  $F$  if for each component  $T$  of  $F$  the edges of  $R \cap E(T)$  lie in a single path.

The next lemma plays a main role in the proofs of the results of this section.

**Lemma 6** Let  $R$  be a linear connector of a forest  $F = \bigcup_{i=1}^m T_i$ . Suppose that  $f$  is a proper labeling of  $F - R$ . Then, there exists a labeling  $g$  of  $F$  such that:

- (i)  $g$  preserves the weights induced by  $f$  on the edges of  $F - R$  and,
- (ii)  $g$  assigns to each of the edges of  $R$  the same weight  $h$ , where  $h$  is none of the weights induced by  $f$ .

**Proof** Without loss of generality, we may assume that for some  $r$ ,  $1 \leq r \leq m$ ,  $t_i = |E(T_i) \cap R| > 0$  if  $1 \leq i \leq r$  and, when  $r < m$ ,  $E(T_i) \cap R = \emptyset$  if  $r < i \leq m$ .

For  $i = 1, 2, \dots, r$ , let  $e_j^{(i)} = v_{2j-1}^{(i)} v_{2j}^{(i)}$ ,  $1 \leq j \leq t_i$ , be the edges in  $E(T_i) \cap R$ . Since the edges  $e_1^{(i)}, e_2^{(i)}, \dots, e_{t_i}^{(i)}$  lie in a single path of  $T_i$ , it follows that the vertices  $v_{2j}^{(i)}$  and  $v_{2j+1}^{(i)}$  belong to the same component of  $F - R$ , for all  $j = 1, 2, \dots, t_i - 1$ . (We remark that it may be possible that  $v_{2j}^{(i)} = v_{2j+1}^{(i)}$ , for some  $i$ ,  $1 \leq i \leq r$ , and for some  $j$ ,  $1 \leq j \leq t_i - 1$ .)

Let  $C_0^{(i)}$  be the component of  $F - R$  that contains the vertex  $v_1^{(i)}$ ,  $1 \leq i \leq r$ , and denote the component of  $F - R$  that contains the vertex  $v_{2j}^{(i)}$  by  $C_j^{(i)}$ ,  $j = 1, 2, \dots, t_i$ . In Figure 7, we depict the component  $T_i$  of  $F$  and the components  $C_j^{(i)}$  of  $F - R$  obtained from  $F$  by the deletion of the edges of  $E(T_i) \cap R$ .

$T_i$ :

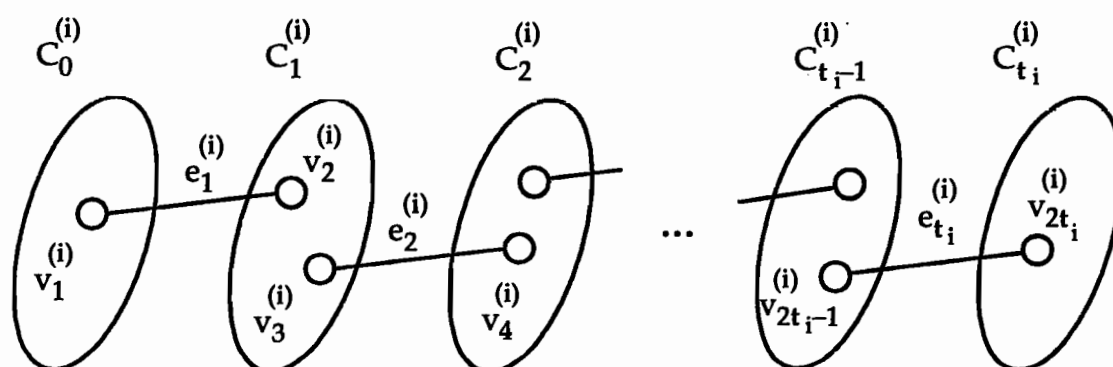


Figure 7

Let  $M = \max \{f(x) \mid x \in V(F)\}$ . As a matter of convenience, we define  $b(e_j^{(i)}) = f(v_{2j-1}^{(i)}) - f(v_{2j}^{(i)})$ , for  $1 \leq j \leq t_i$  and  $1 \leq i \leq r$ . Assume that  $x$  is a vertex of the forest  $F$ . Now, we proceed to define the labeling  $g$ .

*Case 1:*  $x \in V(T_i)$  for some  $i > r$ . In this case, define  $g(x) = f(x)$ .

*Case 2:*  $x \in V(T_i)$  for some  $i$ ,  $1 \leq i \leq r$ . In order to define  $g$ , let

$$b(C_j^{(i)}) = \sum_{z=1}^{i-1} \sum_{y=1}^{t_z} b(e_y^{(z)}) + \sum_{y=1}^j b(e_y^{(i)}), \text{ and}$$

$$d(C_j^{(i)}) = \sum_{z=1}^{i-1} t_z + j,$$

for each  $i$ ,  $1 \leq i \leq r$  and for each  $j$ ,  $0 \leq j \leq t_i$ .

(In the definitions given above, take sums equal to 0 if the upper limit of the sum is less than the corresponding lower limit.)

Note that

$$b(C_j^{(i)}) - b(C_{j-1}^{(i)}) = b(e_j^{(i)}) \text{ and } d(C_j^{(i)}) - d(C_{j-1}^{(i)}) = 1, \quad (1)$$

for each  $1 \leq i \leq r$ ,  $1 \leq j \leq t_i$ ; and

$$b(C_{t_i}^{(i)}) = b(C_0^{(i+1)}) \text{ and } d(C_{t_i}^{(i)}) = d(C_0^{(i+1)}), \quad (2)$$

for each  $1 \leq i \leq r-1$ .

Suppose that  $x \in V(C_j^{(i)})$  for some  $j$ ,  $0 \leq j \leq t_i$ . In this second case, we define the labeling  $g$  as follows.

$$g(x) = f(x) + b(C_j^{(i)}) + 2M d(C_j^{(i)}).$$

The next three observations allow us to prove that  $g$  is one-to-one.

- (i) For given  $i$  and  $j$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq t_i$ , and for each vertex  $x \in V(C_j^{(i)})$  we have that  $g(x) = f(x) + c$ , where  $c$  is a constant.
- (ii) For a given  $i$ ,  $1 \leq i \leq r-1$ , and for each vertex  $x \in V(C_{t_i}^{(i)}) \cup V(C_0^{(i+1)})$ , we have that  $g(x) = f(x) + c'$ , where  $c'$  is a constant. (This follows directly from (2).)

(iii) For given  $i$  and  $j$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq t_i$ , let  $x_{j-1} \in V(C_{j-1}^{(i)})$  and  $x_j \in V(C_j^{(i)})$ . Then,

$$g(x_j) > g(x_{j-1}).$$

To see this, we use (1) to obtain

$$\begin{aligned} g(x_j) - g(x_{j-1}) &= (f(x_j) - f(x_{j-1})) + b(e_j^{(i)}) + 2M \\ &= (f(x_j) - f(x_{j-1})) + (f(v_{2j-1}^{(i)}) - f(v_{2j}^{(i)})) + 2M. \end{aligned} \quad (3)$$

Now we show that  $g(x_j) > g(x_{j-1})$ . Since  $x_{j-1} \in V(C_{j-1}^{(i)})$  and  $v_{2j}^{(i)} \in V(C_j^{(i)})$ , it follows that  $x_{j-1} \neq v_{2j}^{(i)}$ . In a similar manner we obtain that  $x_j \neq v_{2j-1}^{(i)}$ . From equation (3) and since  $f$  is a proper labeling of  $G$ , it follows that if  $x_{j-1} = v_{2j-1}^{(i)}$  or  $x_j = v_{2j}^{(i)}$  then  $g(x_j) - g(x_{j-1}) \geq M$ . Thus, we may assume that  $x_{j-1} \neq v_{2j-1}^{(i)}$  and  $x_j \neq v_{2j}^{(i)}$ . Since  $|f(u) - f(v)| = M$  for at most two vertices  $u, v$  of  $F$ , we conclude that  $g(x_j) - g(x_{j-1}) \geq 1$ .

Observation (i) also shows that  $g$  preserves the weights induced by  $f$  on the edges of  $F - R$ .

We now prove that  $g$  assigns the same weight  $2M$  to each edge of  $R$ . Let  $e \in R$ . Then for some  $i$ ,  $1 \leq i \leq r$ , and for some  $s$ ,  $1 \leq s \leq t_i$ , we have that  $e = e_s^{(i)} = v_{2s-1}^{(i)} v_{2s}^{(i)}$ . Note that  $v_{2s}^{(i)} \in V(C_s^{(i)})$  and  $v_{2s-1}^{(i)} \in V(C_{s-1}^{(i)})$ . Thus, using (1) we obtain that the weight induced by  $g$  on the edge  $e$  is

$$\begin{aligned} &|g(v_{2s}^{(i)}) - g(v_{2s-1}^{(i)})| = \\ &|(f(v_{2s}^{(i)}) + b(C_s^{(i)}) + 2M d(C_s^{(i)})) - (f(v_{2s-1}^{(i)}) + b(C_{s-1}^{(i)}) + 2M d(C_{s-1}^{(i)}))| = \\ &|(b(C_s^{(i)}) - b(C_{s-1}^{(i)}) - b(e_s^{(i)})) + 2M (d(C_s^{(i)}) - d(C_{s-1}^{(i)}))| = 2M. \quad \square \end{aligned}$$

A  $k$ -linear connector of a forest  $F$  is a linear connector of  $F$  that has cardinality  $k$ . Note that if  $F$  has a component whose diameter is  $d$ , then  $F$  has a  $k$ -linear connector for each  $k \leq d$ .

We next prove that every forest of even size is 2-equitable.

**Theorem 7** Every forest of even size is 2-equitable.

**Proof** Let  $F$  be a forest of size  $2n$ ,  $n \geq 1$ . The proof is by induction on  $n$ . If  $n = 1$  then,  $F$  is a linear forest of size 2 and, by Lemma 1,  $F$  is 2-equitable. Assume that each forest of size  $2n$  is 2-equitable. Observe that each forest of size at least 2 has a 2-linear connector. Let  $R$  be a 2-linear connector of a forest  $G$  of size  $2(n + 1)$ . Then, by inductive hypothesis,  $G - R$  is 2-equitable and, by an application of Lemma 6, it follows that  $G$  is 2-equitable.  $\square$

Let  $T$  be a tree of order at least 3 and, suppose that the deletion of all the end-vertices of  $T$  transforms  $T$  into a path  $P$ . Then, the tree  $T$  is called a *caterpillar* and the path  $P$  is called the *spine* of  $T$ . Let  $P: v_1, v_2, \dots, v_n$  be the spine of the caterpillar  $T$  and, for  $i = 1, 2, \dots, n$ , let  $d_i$  denote the number of end-vertices of  $T$  adjacent to  $v_i$ . Since the numbers  $d_1, d_2, \dots, d_n$  characterize  $T$ , we also denote  $T$  by  $S_{d_1, d_2, \dots, d_n}$ . In particular, if  $n = 1$  the caterpillar  $S_{d_1}$  is isomorphic to the star  $K_{1, d_1}$  and, if  $n = 2$  the caterpillar  $S_{d_1, d_2}$  is called a *double star*.

We now show that, for  $k = 3$  and  $k = 4$ , the degree constraint given in Lemma 2 is a sufficient condition for the  $k$ -equitability of a forest whose size is divisible by  $k$ .

**Theorem 8** For  $k = 3$  and for  $k = 4$ , let  $F$  be a forest of size  $q = kw$ . Then, the forest  $F$  is  $k$ -equitable if and only if  $\Delta(F) \leq 2w$ .

**Proof**  $\Rightarrow$ ) The necessity follows from Lemma 2.

$\Leftarrow$ ) Let  $F$  be a forest of size  $kw$  such that  $\Delta(F) \leq 2w$ . We use induction on  $w$  to prove that  $F$  is  $k$ -equitable.

Suppose that  $w = 1$ . Then  $F$  is a linear forest of size  $k$ . By Lemma 1,  $F$  is  $k$ -equitable. Assume that every forest  $F$  of size  $kw$ ,  $w \geq 1$ , for which  $\Delta(F) \leq 2w$  is  $k$ -equitable. Let  $G$  be a forest of size  $q = k(w + 1)$  such that  $\Delta(G) \leq 2(w + 1)$ . To show that  $G$  is  $k$ -equitable it is convenient to divide the rest of the proof into two cases.

*Case 1:*  $k = 3$ . Since  $G$  has size at least 6, it follows that in  $G$  there are at most two vertices of degree at least  $\frac{2q}{3} - 1$  and, at most one of these two vertices may

have degree equal to  $\frac{2q}{3}$ . Now, we proceed to choose a convenient 3-linear connector  $R$  of  $G$ . We consider three subcases.

*Subcase 1.1:* There exist two distinct vertices  $u, v$  in  $G$  that have degree at least  $\frac{2q}{3} - 1$ . Assume that  $\deg u \geq \deg v$ . If  $u$  and  $v$  belong to the same component, let  $u, v_1, v_2, \dots, v_n = v$  be a  $u$ - $v$  path in  $G$ , and let  $e_1 = uv_1$ . We choose  $R = \{e_1, e_2, e_3\}$ , where  $e_2$  and  $e_3$  are two edges different from  $e_1$  that are incident with  $u$  and  $v$ , respectively. If  $u$  and  $v$  belong to distinct components of  $G$  then let  $R = \{e_1, e_2, e_3\}$ , where  $e_1$  and  $e_2$  are two distinct edges incident with  $u$  and  $e_3$  is an edge incident with  $v$ .

*Subcase 1.2:* There exists exactly one vertex  $u$  in  $G$  with degree at least  $\frac{2q}{3} - 1$ . If there is a path  $P$  of length 3 in  $G$  that contains  $u$  as one of its central vertices then, take  $R = E(P)$ . Otherwise, the component  $C$  of  $u$  is a star of size at most  $\frac{2q}{3}$  and  $G$  must have another component  $C'$  with positive size. Let  $e \in E(C')$  and let  $P$  be a path of length 2 in  $C$ . Then, we choose  $R = E(P) \cup \{e\}$ .

*Subcase 1.3:*  $\Delta(G) \leq \frac{2q}{3} - 2$ . It is straightforward to see that  $G$  has a 3-linear connector. In this subcase we choose any 3-linear connector  $R$  of  $G$ .

Observe that in each of these three subcases the forest  $F = G - R$  has size  $kw$  and  $\Delta(F) \leq \frac{2q}{3} - 2 = 2w$ . Therefore, by inductive hypothesis,  $F$  is 3-equitable. By an application of Lemma 6 we obtain that  $G$  is 3-equitable.

*Case 2:*  $k = 4$ . First, we observe that if  $G$  has more than two vertices of degree at least  $\frac{q}{2} - 1$ , then  $q = 8$  and  $G$  is isomorphic to one of the seven forests of Figure 8. It is straightforward to see that these forests are 4-equitable. Therefore, we may assume that  $G$  has at most two vertices of degree at least  $\frac{q}{2} - 1$ . We now proceed to choose a 4-linear connector  $R$  of  $G$ . Again, we consider three subcases.

*Subcase 2.1: There exist two distinct vertices  $u, v$  in  $G$  that have degree at least  $\frac{q}{2} - 1$ .*

Assume that  $u$  and  $v$  belong to the same component  $C$  of  $G$ . If there is a path  $P$  of length 4 that contains  $u$  and  $v$  as internal vertices then, choose  $R = E(P)$ . Otherwise, the component  $C$  is isomorphic to a double star of size at most  $q - 1$  and  $G$  must have another component  $C'$  with positive size. Let  $e \in E(C')$  and let  $P$  be a path of length 3 in  $C$ . Then, we choose  $R = E(P) \cup \{e\}$ .

If  $u$  and  $v$  belong to distinct components of  $G$  then, let  $P_u$  and  $P_v$  be two paths of length 2 that have  $u$  and  $v$  as their corresponding central vertices. We choose  $R = E(P_u) \cup E(P_v)$ .

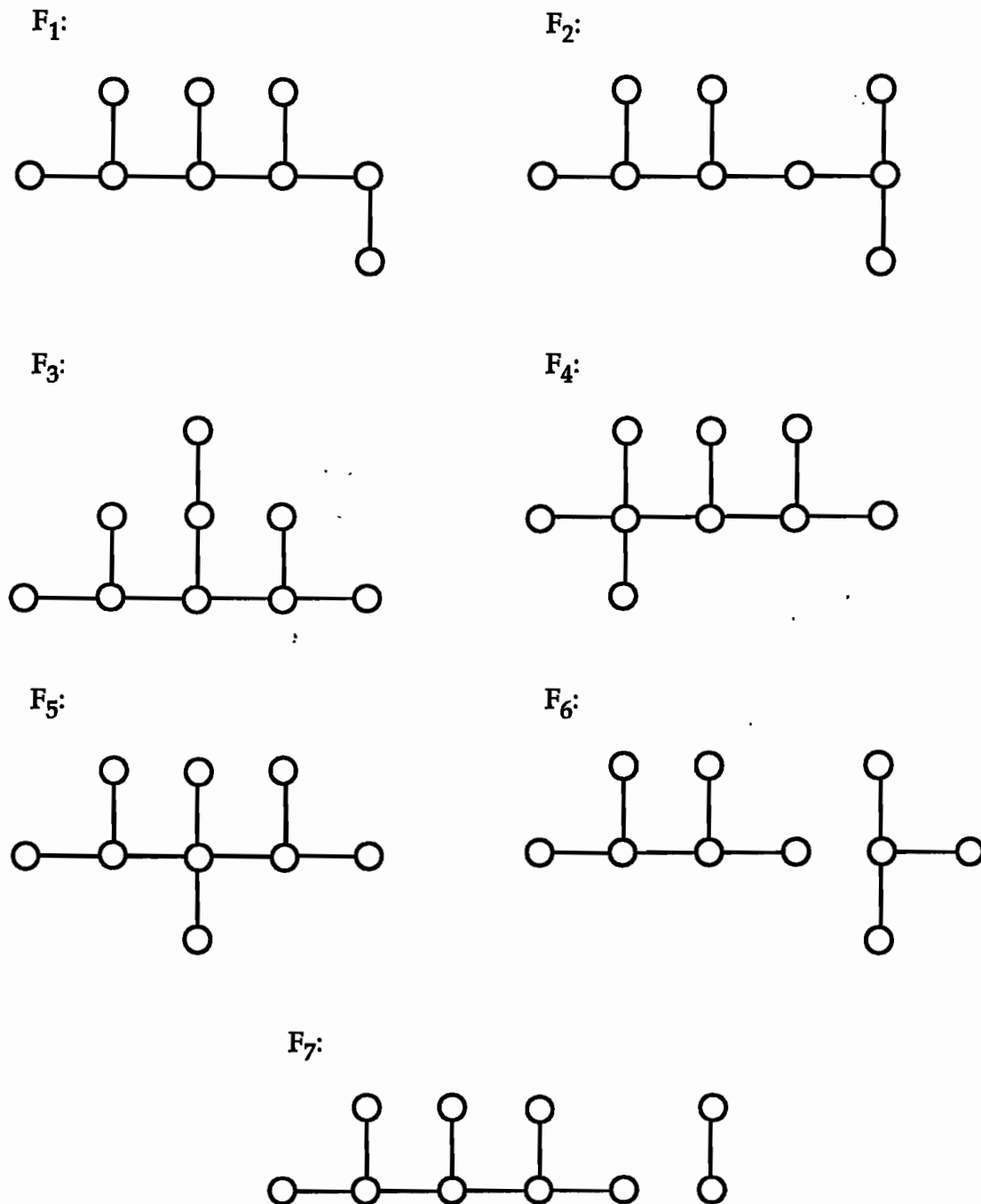


Figure 8

*Subcase 2.2: There exists exactly one vertex  $u$  in  $G$  with degree at least  $\frac{q}{2} - 1$ . Assume that there is a path  $P$  of length 4 in  $G$  that contains  $u$  as one of its*

internal vertices. Then, we choose  $R = E(P)$ . Otherwise, the component  $C$  of  $u$  is a tree of size at most  $q - 3$  and we can easily find in  $G$  a 4-linear connector  $R$  that contains two edges incident with  $u$ .

*Subcase 2.3:*  $\Delta(G) \leq \frac{q}{2} - 2$ . It is straightforward to see that  $G$  has a 4-linear connector. In this subcase we choose any 4-linear connector  $R$  of  $G$ .

Similarly to Case 1, we observe that in each of these three subcases the forest  $F = G - R$  has size  $kw$  and  $\Delta(F) \leq \frac{q}{2} - 2 = 2w$ . Therefore, by inductive hypothesis,  $F$  is 4-equitable. By an application of Lemma 6 we obtain that  $G$  is 4-equitable.  $\square$

**Corollary 9** For  $k = 3$  and  $k = 4$ , let  $F$  be a forest of size  $q = kw$ . Then,  $F$  can be decomposed into  $w$  linear forests of size  $k$  if and only if  $\Delta(F) \leq 2w$ .

Two observations related to Lemma 6 now follow.

Lemma 6 may be used to prove the  $k$ -equitability of certain classes of graphs. For example, we can construct a tree  $T$  from  $m + 1$  paths  $P_{n_i}$ ,  $1 \leq i \leq m$ ,  $m \geq 1$ , by choosing vertices  $u^{(n_i)} \in V(P_{n_i})$  and  $v^{(n_{i+1})} \in V(P_{n_{i+1}})$ , for each  $i = 1, 2, \dots, m$ , which are then correspondingly joined by the edges  $u^{(n_i)} v^{(n_{i+1})}$ . We call the tree  $T$  a *fence* of width  $m$ . (In Figure 9, we show a fence whose width is 5.) From Lemma 6 and Corollary 4, we obtain the following result.

**Theorem 10** Let  $T$  be a fence of width  $k$  whose size is divisible by  $k$ . Then,  $T$  is  $k$ -equitable.

Note that the fence shown in Figure 9 is 5-equitable. Theorem 10 also shows the existence of  $k$ -equitable trees that are not isomorphic to paths, for each positive integer  $k$ .

Lemma 6 can be generalized to an arbitrary graph  $G$ , extending the definition of a linear connector  $R$  to a set of bridges of  $G$  such that, for each component  $C$  of  $G$ , the edges of  $R \cap E(C)$  lie in a single path.

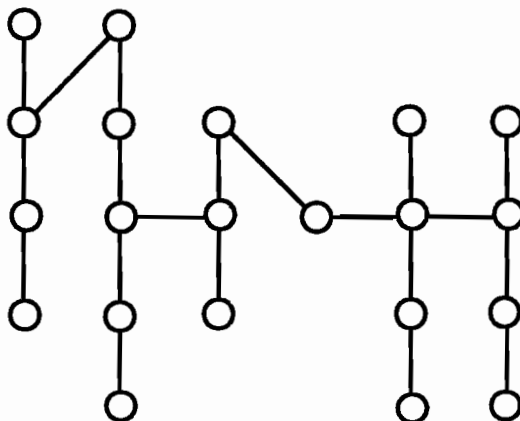


Figure 9

#### 4. On the $k$ -equitability of caterpillars, $k \geq 5$ .

For  $k \geq 5$ , the degree constraint given in Lemma 2 is not a sufficient condition to assure the  $k$ -equitability of a forest whose size is divisible by  $k$ . For example, the caterpillar  $T$  of size 10 shown in Figure 10 is not 5-equitable (see Lemma 11); nevertheless,  $T$  satisfies the degree constraint  $\Delta(T) \leq 2w = 4$ .

$T$ :

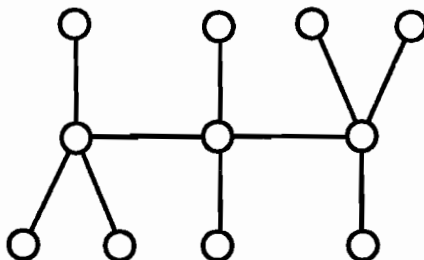


Figure 10

Let  $f$  be a  $k$ -equitable labeling of a graph  $G$ . We already know that the subgraph  $S_h$  induced by the edges of  $G$  with weight  $h$  is a linear forest. For each edge  $e \in E(G)$ , define  $q_L(e)$  as the cardinality of a largest set of edges of  $G$  that induces a linear forest that contains  $e$ . Then,

$$k \leq \min_{e \in E(G)} \{q_L(e)\}.$$

In particular, if  $T$  is a caterpillar whose spine has order  $n \geq 2$  then  $q_L(e) \leq 2n - 1$ , for each edge  $e$  that belongs to the spine of  $T$ . Thus, in this case, we obtain

$$k \leq 2n - 1.$$

With respect to this last inequality, Joseph McCanna observed (personal communication to one of us) that if  $n \geq 3$  then  $k < 2n - 1$ .

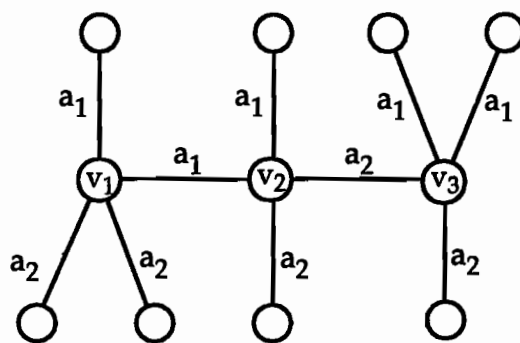
We state these results on caterpillars in the following lemma.

**Lemma 11** Let  $T$  be a  $k$ -equitable caterpillar whose spine has order  $n \geq 2$ . Then,

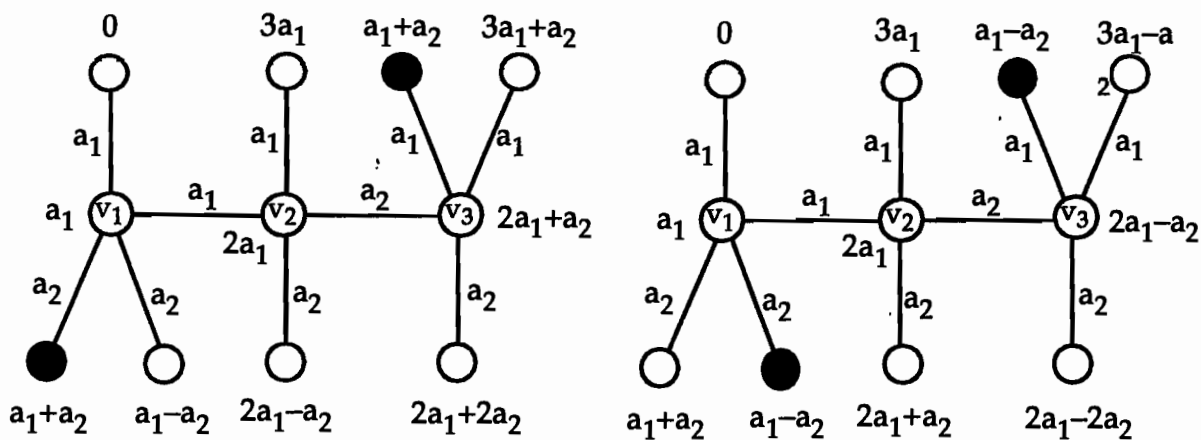
$$k \leq \begin{cases} 3 & \text{if } n = 2 \\ 2n - 2 & \text{if } n \geq 3. \end{cases}$$

**Proof** We only have to prove that if  $n \geq 3$  then,  $k \neq 2n - 1$ . Let  $f$  be a  $k$ -equitable labeling of  $T$  and let  $n \geq 3$ . Suppose, to the contrary, that  $k = 2n - 1$ . Since  $n \geq 3$ , there are two adjacent edges  $e_1$  and  $e_2$  in the spine of  $T$  whose induced weights are different. For  $i = 1, 2$ , let  $e_i = v_i v_{i+1}$  and denote the weight of the edge  $e_i$  by  $a_i$ . Since  $k = 2n - 1$ , it follows that for  $i = 1, 2$  the linear forest induced by the edges that have weight  $a_i$  is isomorphic to  $(n - 2)P_3 \cup P_4$ , where  $e_i$  is the central edge of the component isomorphic to  $P_4$ . The subgraph  $H$  of  $T$  induced by the edges of weight  $a_1$  or  $a_2$  incident with some of the vertices  $v_1, v_2$ , or,  $v_3$  is shown in Figure 10 (a). It is now straightforward to see that the distribution of weights shown in Figure 10 (a) forces the labels of two distinct vertices of  $H$  to be equal, which is a contradiction. (See the two basic cases in Figures 11 (b) and (c).)  $\square$

H:



(a)



(b)

(c)

Figure 11

Let  $W$  be the set of the weights induced by a  $k$ -equitable graph  $G$  and let  $S_h$  be the subgraph of  $G$  induced by the edges of  $G$  that have weight  $h$ . Then, the collection of subgraphs  $\{S_h \mid h \in W\}$  is a decomposition of  $G$  into linear forests of size  $k$ , called a *common weight decomposition* of  $G$  (see [3]). Thus, each  $k$ -equitable labeling of  $G$  determines a decomposition of  $G$  into linear forests of equal size. The caterpillar  $T$  in Figure 9 shows that the existence of such a decomposition of  $G$  does not guarantee the existence of a corresponding  $k$ -equitable labeling of  $G$ .

With respect to decompositions of caterpillars into linear forests of equal size, we believe that the following conjecture is true.

**Conjecture** Let  $T$  be a caterpillar of size  $kw$ . Then,  $T$  has a decomposition into  $w$  linear forests of size  $k$  if and only if  $\Delta(G) \leq 2w$ .

## 5. Equitable labelings of caterpillars

In this section we show the existence of some optimal and complete  $k$ -equitable labelings of caterpillars.

**Theorem 12** Let  $T$  be a caterpillar of even size  $q$ . Then, there exists an optimal complete 2-equitable labeling  $f$  of  $T$ .

**Proof** The construction of  $f$  is a generalization of the well-known construction given by Rosa in [9] to prove that caterpillars are graceful. First, we find two connected edge-disjoint subgraphs  $T_1$  and  $T_2$  of  $T$ , each having size  $\frac{q}{2}$ . Let  $v$  be the unique common vertex of  $T_1$  and  $T_2$ . Note that for  $i = 1, 2$ , the vertex  $v$  is an end-vertex of the spine of  $T_i$  or  $v$  is adjacent to such a vertex. In any case, using Rosa's construction, we may find a graceful labeling  $f_i$  of  $T_i$ ,  $i = 1, 2$ , for which  $f_1(v) = \frac{q}{2}$  and  $f_2(v) = 0$ . We define the labeling  $f$  of  $T$  as follows.

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in V(T_1) \\ f_2(x) + \frac{q}{2} & \text{if } x \in V(T_2). \quad \square \end{cases}$$

Following the procedure given in the proof of Theorem 12, in Figure 12 we show some steps in the construction of an optimal complete 2-equitable labeling  $f$  of a caterpillar  $T$  of even size.

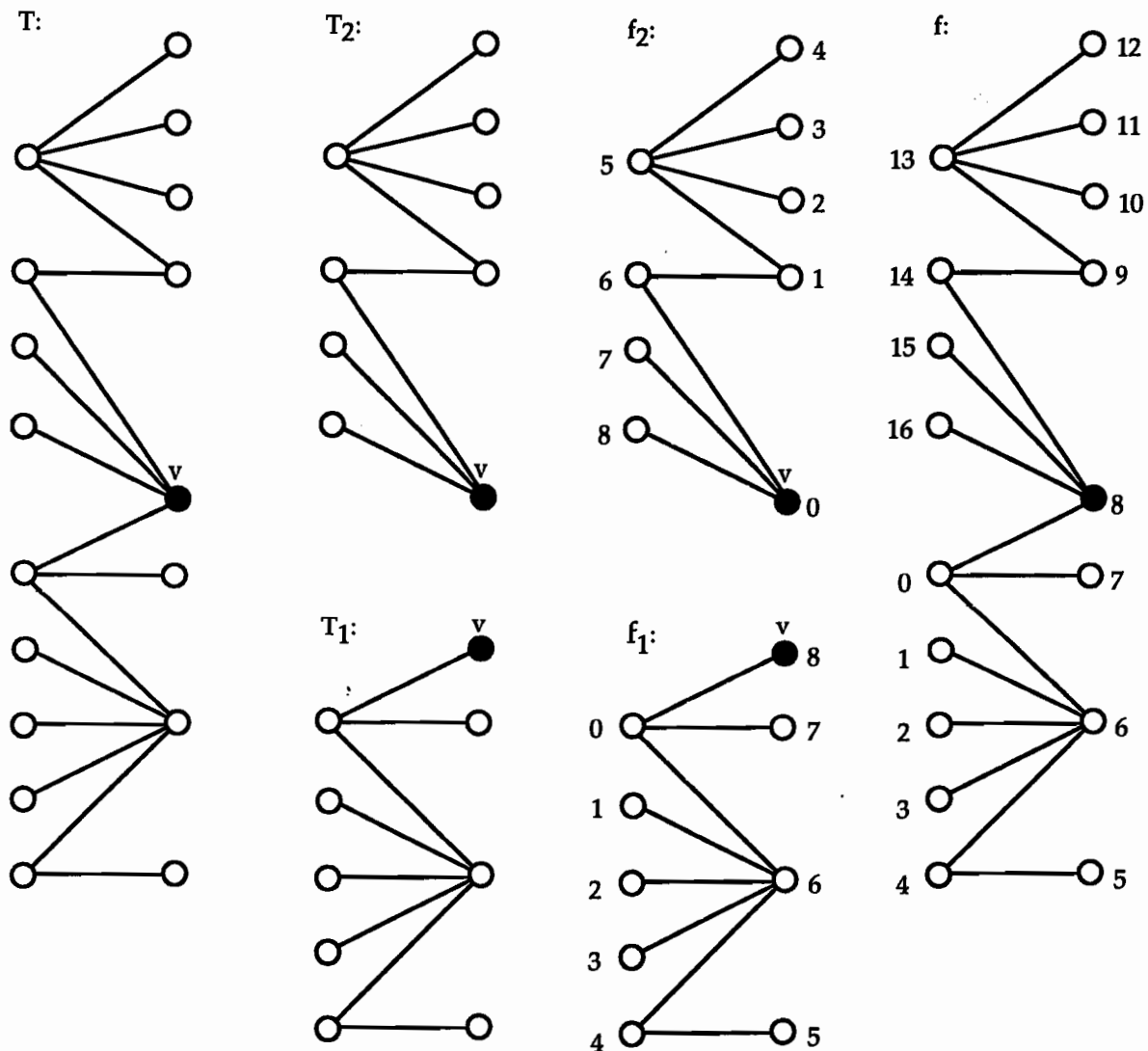


Figure 12

Let  $T$  be a  $k$ -equitable caterpillar. If the spine of  $T$  has order 1 (that is, if  $T \cong K_{1,n}$ ,  $n \geq 2$ ) then, by Lemma 2,  $k \leq 2$  and, for these two possible values of  $k$ , there exist  $k$ -equitable labelings of  $T$  that are optimal and complete. (See Theorem 12.)

If the spine of a  $k$ -equitable caterpillar  $T$  has order 2 (that is, if  $T$  is a double star) then, by Lemma 10,  $k \leq 3$ . For  $k = 2$ , again Theorem 12 assures the existence of an optimal complete 2-equitable labeling of  $T$ . If  $k = 3$ , it is not true in general that  $T$  has a complete 3-equitable labeling. In Figure 13 we show the smallest 3-equitable double star that does not possess a complete 3-equitable labeling.

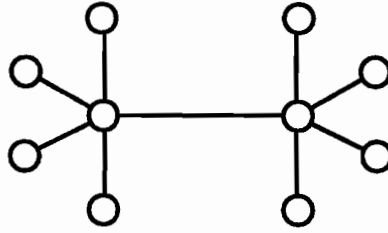


Figure 13

The following lemma will be used to describe a 3-equitable labeling of each 3-equitable double star.

**Lemma 13** Let  $q$  be the size of the double star  $S_{m,n}$ ,  $1 \leq m \leq n$ , and assume that 3 divides  $q$ . Then,  $S_{m,n}$  is 3-equitable if and only if there exists an integer  $s$ ,  $0 \leq s \leq \frac{q-3}{6}$ , such that  $m = \frac{q}{3} + s$  and  $n = \frac{2q}{3} - s - 1$ .

**Proof** Let  $C = S_{m,n}$ . Denote the vertices of the spine of  $C$  by  $u$  and  $v$ , where  $\deg u = m + 1$  and  $\deg v = n + 1$ .

$\Rightarrow$ ) Suppose that  $C$  is 3-equitable. Let  $\mathcal{D}$  be a common weight decomposition of  $C$  induced by a 3-equitable labeling of  $C$ . Observe that the graph  $D$  of  $\mathcal{D}$  that contains the unique edge of the spine of  $C$  must be isomorphic to  $P_4$ . On the other hand, each of the graphs in  $\mathcal{D} - \{D\}$  is isomorphic to  $P_2 \cup P_3$ . Note that for each  $H \in \mathcal{D} - \{D\}$ , the vertex of degree 2 of  $H$  is either  $u$  or  $v$ . Assume that there are exactly  $s$  graphs in  $\mathcal{D} - \{D\}$  for which  $u$  is the vertex of degree 2, and suppose that there are exactly  $t$  graphs in  $\mathcal{D} - \{D\}$  for which  $v$  is the vertex of degree 2. Then,

$$m = 2s + t + 1 \quad \text{and} \quad n = s + 2t + 1. \quad (4)$$

Since the number of graphs of  $\mathcal{D}$  is  $\frac{q}{3}$ , we have

$$\frac{q}{3} = s + t + 1. \quad (5)$$

Combining (4) and (5) we obtain

$$m = \frac{q}{3} + s \quad \text{and} \quad n = \frac{2q}{3} - s - 1,$$

and from  $m \leq n$ , we deduce that  $0 \leq s \leq \frac{q-3}{6}$ .

$\Leftarrow$ ) Clearly,  $C$  satisfies the degree constraint. Therefore, by Theorem 7,  $C$  is 3-equitable.  $\square$

Let  $q$  be the size of a 3-equitable double star  $C = S_{m, n}$ ,  $1 \leq m \leq n$ . By Lemma 13, there exists an integer  $s$ ,  $0 \leq s \leq \frac{q-3}{6}$ , such that  $m = \frac{q}{3} + s$  and  $n = \frac{2q}{3} - s - 1$ . In Figure 14, where  $\deg u = m$  and  $\deg v = n$ , we describe a 3-equitable labeling of  $C$  whose weights constitute the set  $\{1, 3, \dots, \frac{2q}{3} - 1\}$ . (When  $s = 0$ , the bold vertices and their incident edges should be omitted from this figure.)

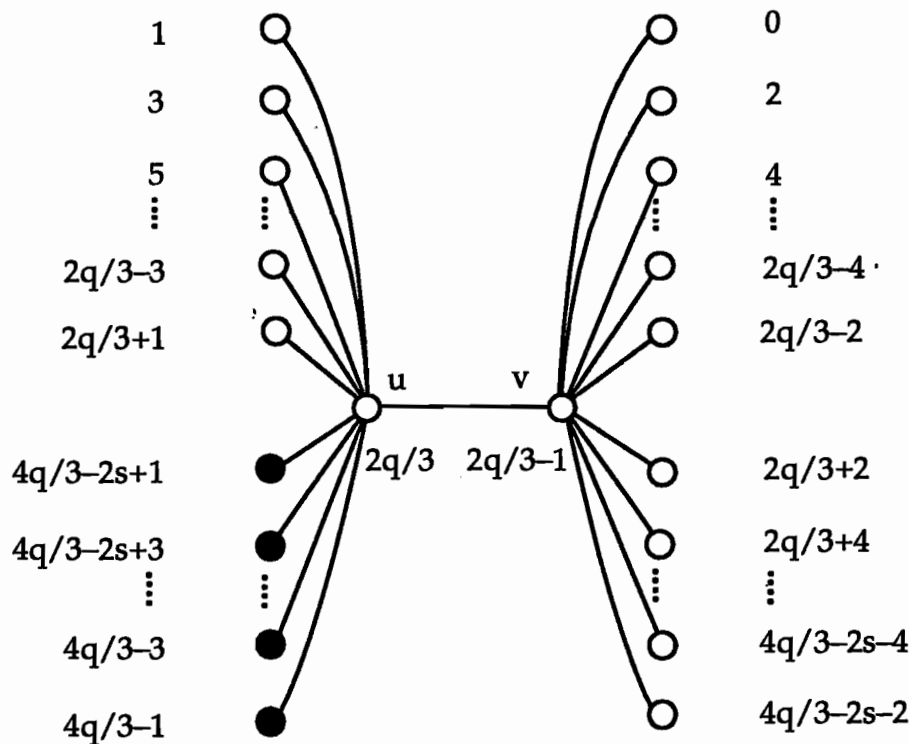


Figure 14

Using Lemma 13, we easily obtain the following characterization.

**Theorem 14** Assume that 3 divides the size  $q$  of the double star  $S_{m, n}$ ,  $1 \leq m \leq n$ . Then,  $S_{m, n}$  is 3-equitable if and only if  $\frac{q}{3} \leq m \leq \lfloor \frac{q-1}{2} \rfloor$ .

In particular, we know of an infinite family of double stars that possesses optimal complete 3-equitable labelings.

**Theorem 15** Let  $m \geq 1$ . Then, there exists an optimal complete 3-equitable labeling of the double star  $S_{m, 2m-1}$ .

**Proof** Let  $C = S_{m, 2m-1}$ . The result is clear if  $m = 1$ . Let  $m \geq 2$ . It is straightforward to see that the labeling of  $C$  given in Figure 15 satisfies the required conditions. (In Figure 15,  $\deg u = m + 1$  and  $\deg v = 2m$ .)  $\square$

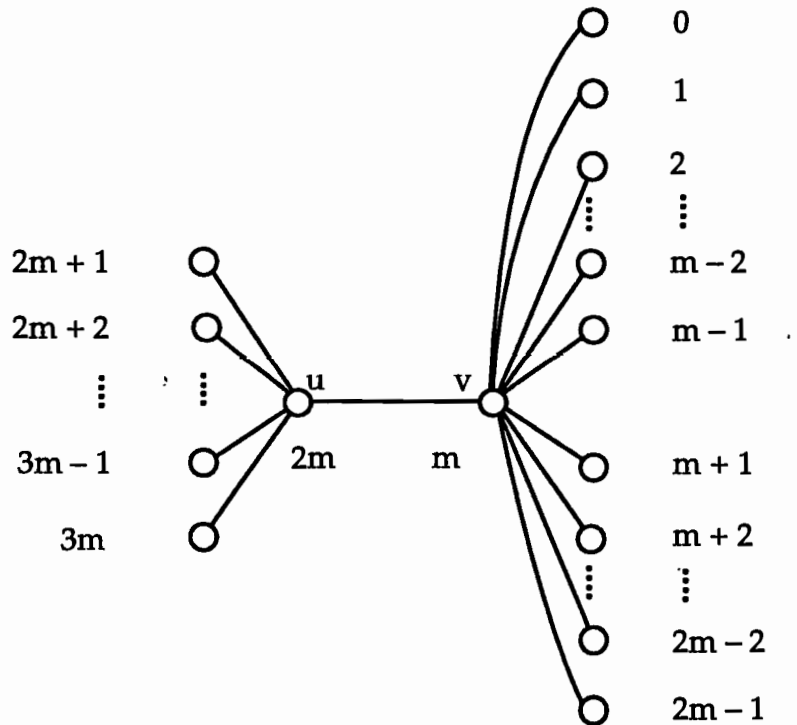


Figure 15

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